Predicting Layer Cracks in Cross-Laminated Timber with Evaluations of Strategies for Suppressing Them

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Electronic Supplementary Materials

The printed paper associated with this on-line resourse contains a complete derivation of energy release rate due to layer cracks for a cross-laminated timber (CLT) panel subjected to in-plane mechanical and residual stresses. While the analysis in the paper is complete, some details were omitted for brevity. This supplemental information repeats the derivation with additional details. Although a scientist with a background in mechanics could repeat the paper derivation, the intent of this resource is to make that process easier. Besides derivation details, this information adds some other details about derivations and calculation results. The topics covered are:

1. Panel Cracking Energy Release Rate Derivation: Repeats the derivation in the paper for energy release rate due to layer cracking but fills in omitted details.
2. Uncracked Laminate Properties: Derives properties of CLT with no cracks from laminated plate theory.
3. Generalized and Simplified Analysis for Tensile Moduli: Generalizes the analyses in Hashin (1987) and Nairn (2017) to apply to panels in which layer and surface layers are different and simplifies the final results.
4. Drying of Surface Layers Compared to Dying All Layers: More detailed calculation of stresses in CLT panels prepared by drying core and surface layers by different amounts prior to fabrication of the panel.
5. Onset of Delamination: Explains how the paper determined a dimensionless crack density at onset of delamination to be $1/\rho = 0.55$ even though that number never appears in the cited reference.

1 Panel Cracking Energy Release Rate Derivation

Figure 1 shows a unit cell for a three-layer CLT panel with orthogonal cracks in all layers (Hashin, 1987). The core layer (referred to here as layer 1) has thickness $2t_1$, cracks on either end at $x = \pm a$, and wood grain at 90° to the 1 direction (the core layer is therefore also referred to as the 90° layer). The surface layers (referred to here as layer 2) have thickness t_2, cracks on either end at $y = \pm b$, and wood grain at 0° to the 1 direction (the surface layers are therefore also referred to as the 0° layers). The total thickness is $2h = 2(t_1 + t_2)$. Repeating this unit cell leads to panel with periodic cracks having crack density $D_a = 1/(2a)$ in the core layer and $D_b = 1/(2b)$ in the surface layers. These are dimensioned crack density or cracks per unit length in each layer. Once stress an energy are found for the init cell, the energy in a full CLT panel is found by averaging a collection of unit cells of varying sizes depending on statistical distributions of crack spacings ($2a$ and $2b$) on the panel edges.

The energy release rate due to any increment in fracture area, dA, within a composite with residual thermal stresses is derived in Nairn (1997) as:

$$G = G_{mech} + \frac{V}{2}\left(2\frac{d\left\langle \sigma^m \cdot \alpha\Delta T\right\rangle}{dA} + \frac{d\left\langle \sigma^r \cdot \alpha\Delta T\right\rangle}{dA}\right) \tag{1}$$

where G_{mech} is energy released due to mechanical stresses alone, V is total volume, total stress is partitioned into mechanical and residual stresses $\sigma = \sigma^m + \sigma^r$, α is thermal expansion tensor, ΔT is difference between current temperature and the stress-free temperature, and angle brackets indicate a volume-averaged quantity:

$$\langle f(x,y,z)\rangle = \frac{1}{V}\int_V f(x,y,z)dV \tag{2}$$

For in-plane loading, all through-the-thickness (z direction) stress averages will be zero. The two averages needed here can therefore be rewritten as:

$$\langle \sigma^m \cdot \alpha\Delta T\rangle = \frac{1}{V}\left[\int_{v_1}\left(\sigma_{xx}^{(m)}\alpha_{xx}^{(1)} + \sigma_{yy}^{(m)}\alpha_{yy}^{(1)}\right)\Delta T\,dV + \int_{v_2}\left(\sigma_{xx}^{(m)}\alpha_{xx}^{(2)} + \sigma_{yy}^{(m)}\alpha_{yy}^{(2)}\right)\Delta T\,dV\right] \tag{3}$$

$$= \sum_{k=1}^{2} V_k\left(\overline{\sigma_{xx}^{(km)}}\alpha_{xx}^{(k)} + \overline{\sigma_{yy}^{(km)}}\alpha_{yy}^{(k)}\right)\Delta T \tag{4}$$

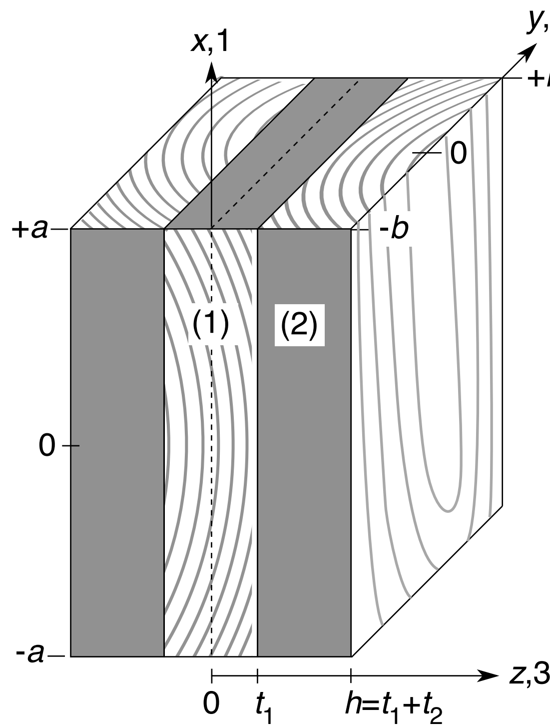

Fig. 1 Unit cell for a CLT panel with orthogonal cracking in all layers. The crack surfaces (which are on both ends) are indicated by gray planes transverse to wood grain directions in those layers. Layer 1 is core layer with thickness t_1 with cracks separated by $2a$. Layers 2 are surface layers with thickness t_2 with cracks separated by $2b$. Axes $(1,2,3)$ are used to refer to panel properties while (x,y,z) refer to layer properties.

where v_k is dimensioned volume of phase k and $V_k = v_k/V$ is volume fraction of phase k. The over-bar indicates a quantity averaged over volume of one phase such as

$$\overline{\sigma_{xx}^{(km)}} = \frac{1}{v_k}\int_{v_k} \sigma_{xx}^{(m)}\, dV \tag{5}$$

The simplification to phase average stresses is possible because within each phase, the thermal expansion coefficients, $\alpha_{xx}^{(k)}$ and $\alpha_{yy}^{(k)}$, are constants that can be removed from the integrals. Likewise, the residual stress average term can be rewritten as:

$$\langle \sigma^r \cdot \alpha \Delta T\rangle = \sum_{k=1}^{2} V_k \left(\overline{\sigma_{xx}^{(kr)}}\alpha_{xx}^{(k)} + \overline{\sigma_{yy}^{(kr)}}\alpha_{yy}^{(k)}\right)\Delta T \tag{6}$$

Equation (1) is an exact result for any failure mode. This supplemental information uses it to derive an exact result for energy release rate due to formation of cracks in layers of a CLT panel caused by applied normal stresses σ_{10} and σ_{20} in the 1 and 2 directions and biaxial residual stresses in the x-y plane. For simplicity, the derivation considers only residual thermal stresses, but the final form is easily adjusted to acccount for a combination of thermal and moisture induced residual stress.

The mechanical energy release rate is derived in (Nairn, 1997) as:

$$G_{mech} = \frac{d}{dA}\left(\frac{1}{2}\int_{S_T} \mathbf{T}^0 \cdot \mathbf{u}^m dS - \frac{1}{2}\int_{S_u} \mathbf{T}^m \cdot \mathbf{u}^0 dS\right) \tag{7}$$

where $\mathbf{T}_0$ is applied traction over loaded surface S_T and $\mathbf{u}_0$ is applied displacement over fixed surface S_u. In addition, $\mathbf{u}^m$ is surface displacement on S_T and $\mathbf{T}^m$ is surface traction on S_u; each of these are due to mechanical loading only.

To find G_{mech}, treat the uncracked surfaces as S_T with traction loads $\pm h\sigma_{10}/t_2$ on the uncracked ends in layer 2 and $\pm h\sigma_{20}/t_1$ on the uncracked ends in layer 1. In other words, surface tractions on the unit cell are:

$$\mathbf{T}_0 = \begin{cases} \left(\frac{h\sigma_{10}}{t_2},0,0\right) & \text{for } x=a,\ -b<y<b,\ t_1<|z|<h \\ \left(-\frac{h\sigma_{10}}{t_2},0,0\right) & \text{for } x=-a,\ -b<y<b,\ t_1<|z|<h \\ \left(0,\frac{h\sigma_{20}}{t_1},0\right) & \text{for } y=b,\ -a<x<a,\ -t_1<z<t_1 \\ \left(0,-\frac{h\sigma_{20}}{t_1},0\right) & \text{for } y=-b,\ -a<x<a,\ -t_1<z<t_1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

From global mechanical panel strains in the two directions ($\varepsilon_{10}^{(m)}$ and $\varepsilon_{20}^{(m)}$) and taking the center of the panel as the origin, end displacements on the uncracked ends are:

$$\mathbf{u}^m = \begin{cases} \left(a\varepsilon_{10}^{(m)},0,0\right) & \text{for } x=a,\ -b<y<b,\ t_1<|z|<h \\ \left(-a\varepsilon_{10}^{(m)},0,0\right) & \text{for } x=-a,\ -b<y<b,\ t_1<|z|<h \\ \left(0,b\varepsilon_{20}^{(m)},0\right) & \text{for } y=b,\ -a<x<a,\ -t_1<z<t_1 \\ \left(0,-b\varepsilon_{20}^{(m)},0\right) & \text{for } y=-b,\ -a<x<a,\ -t_1<z<t_1 \end{cases} \tag{9}$$

Integrating $\mathbf{T}_0\cdot\mathbf{u}^m$ over the entire surface (*i.e.*, treating S_u as empty or unit cell having no displacement boundary conditions), the mechanical energy release rate becomes

$$G_{mech} = \frac{1}{2}\frac{d}{dA}\left(4\int_{t_1}^{h}dz\int_{-b}^{+b}dy\,\frac{h\sigma_{10}}{t_2}a\varepsilon_{10}^{(m)} + 2\int_{-t_1}^{t_1}dz\int_{-a}^{+a}dx\,\frac{h\sigma_{20}}{t_1}b\varepsilon_{20}^{(m)}\right) = \frac{V}{2}\frac{d}{dA}\left(\sigma_{10}\varepsilon_{10}^{(m)}+\sigma_{20}\varepsilon_{20}^{(m)}\right) \tag{10}$$

where $V = 8hab$ is unit cell volume. The panel strains that result from mechanical loads can be expressed in terms of *effective* panel moduli as:

$$\varepsilon_{10}^{(m)} = \frac{\sigma_{10}}{E_{11}(a,b)} - \frac{\nu_{12}\sigma_{20}}{E_{11}(a,b)} \qquad \text{and} \qquad \varepsilon_{20}^{(m)} = -\frac{\nu_{12}(a,b)\sigma_{10}}{E_{11}(a,b)} + \frac{\sigma_{20}}{E_{22}(a,b)} \tag{11}$$

where $E_{11}(a,b)$ and $E_{22}(a,b)$ are tensile moduli and $\nu_{12}(a,b)$ is Poisson's ratio for a CLT panel with cracks spacings a and b in layers 1 and 2, respectively. Substituting these strains into Eq. (10) leads to:

$$G_{mech} = \frac{V}{2}\frac{d}{dA}\left(\frac{\sigma_{10}^2}{E_{11}(a,b)} - \frac{2\nu_{12}(a,b)\sigma_{10}\sigma_{20}}{E_{11}(a,b)} + \frac{\sigma_{20}^2}{E_{22}(a,b)}\right) \tag{12}$$

The next steps are to evaluate the $\langle\sigma^m\cdot\alpha\Delta T\rangle$ and $\langle\sigma^r\cdot\alpha\Delta T\rangle$ terms in Eq. (1) and to express them also in terms of effective panel mechanical properties ($E_{11}(a,b)$, $E_{22}(a,b)$, and $\nu_{12}(a,b)$). The needed derivations start with the Levin (1967) equation that takes the following form for CLT:

$$\sigma^m\cdot\alpha(a,b) = \sum_{k=1}^{2}V_k\overline{\sigma^{(km)}}\cdot\alpha^{(k)} = \sum_{k=1}^{2}V_k\left(\overline{\sigma_{xx}^{(km)}}\alpha_{xx}^{(k)} + \overline{\sigma_{yy}^{(km)}}\alpha_{yy}^{(k)}\right) \tag{13}$$

where σ^m is *any* applied mechanical stress, V_k is volume fraction of layer k (which is $k=1$ for core layer and $k=2$ for both surface layers), and the phase average stresses are averaging mechanical stresses due to applied σ^m. The last form is specific for CLT with through-the-thickness average stresses being zero. Comparing to Eq. (4), the Levin (1967) equation with σ^m given by in-plane stresses σ_{10} and σ_{20} directly gives the first term or

$$\langle\sigma^m\cdot\alpha\Delta T\rangle = \sigma^m\cdot\alpha(a,b)\Delta T = \sigma_{10}\alpha_{11}(a,b)\Delta T + \sigma_{20}\alpha_{22}(a,b)\Delta T \tag{14}$$

which again uses zero applied stress in the z direction and $\alpha_{11}(a,b)$ and $\alpha_{22}(a,b)$ are thermal expansion coefficients (CTE) for a CLT panel with cracks spacings a and b in layers 1 and 2, respectively. We can eliminate panel CTEs by a second use of the Levin (1967) equation to find effective CTE of a composite material in terms of the panel's effective mechanical properties. First consider mechanical loading in direction 1 only. The Levin (1967) equation now becomes

$$\sigma_{10}\alpha_{11}(a,b)\Delta T = \sum_{k=1}^{2} V_k \left(\overline{\sigma_{xx}^{(km)}}\alpha_{xx}^{(k)} + \overline{\sigma_{yy}^{(km)}}\alpha_{yy}^{(k)}\right)\Delta T \qquad (15)$$

where these phase average stresses refer to uniaxial loading. For a two-phase composite (0° and 90° layers), we can evaluate total phase average stresses from these six global relations:

$$\begin{aligned}
\sigma_{10} &= V_1\overline{\sigma_{xx}^{(1)}} + V_2\overline{\sigma_{xx}^{(2)}} \\
0 &= V_1\overline{\sigma_{yy}^{(1)}} + V_2\overline{\sigma_{yy}^{(2)}} \\
\overline{\sigma_{xx}^{(1)}} &= Q_{xx}^{(1)}(\overline{\varepsilon_{xx}^{(1)}} - \alpha_{xx}^{(1)}\Delta T) + Q_{xy}^{(1)}(\overline{\varepsilon_{yy}^{(1)}} - \alpha_{yy}^{(1)}\Delta T) \\
\overline{\sigma_{yy}^{(1)}} &= Q_{xy}^{(1)}(\overline{\varepsilon_{xx}^{(1)}} - \alpha_{xx}^{(1)}\Delta T) + Q_{yy}^{(1)}(\overline{\varepsilon_{yy}^{(1)}} - \alpha_{yy}^{(1)}\Delta T) \\
\overline{\sigma_{xx}^{(2)}} &= Q_{xx}^{(2)}(\overline{\varepsilon_{xx}^{(2)}} - \alpha_{xx}^{(2)}\Delta T) + Q_{xy}^{(2)}(\overline{\varepsilon_{yy}^{(2)}} - \alpha_{yy}^{(2)}\Delta T) \\
\overline{\sigma_{yy}^{(2)}} &= Q_{xy}^{(2)}(\overline{\varepsilon_{xx}^{(2)}} - \alpha_{xx}^{(2)}\Delta T) + Q_{yy}^{(1)}(\overline{\varepsilon_{yy}^{(2)}} - \alpha_{yy}^{(2)}\Delta T)
\end{aligned} \qquad (16)$$

The first two equations are force balance in the two panel directions when loaded only in direction 1. The next four relate phase average stresses to phase average strains where $Q_{ij}^{(k)}$ are elements of the laminated plate theory stiffness tensor for layer k:

$$\begin{pmatrix} \sigma_{xx}^{(k)} \\ \sigma_{yy}^{(k)} \\ \sigma_{xy}^{(k)} \end{pmatrix} = \begin{pmatrix} Q_{xx}^{(k)} & Q_{xy}^{(k)} & 0 \\ Q_{xy}^{(k)} & Q_{yy}^{(k)} & 0 \\ 0 & 0 & G_{xy}^{(k)} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} - \alpha_{xx}^{(k)}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}^{(k)}\Delta T \\ \gamma_{xy} \end{pmatrix}, \quad Q_{ii}^{(k)} = \frac{E_{ii}^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}}, \quad \text{and} \quad Q_{xy}^{(k)} = \frac{E_{xx}^{(k)}\nu_{yx}^{(k)}}{1-\nu_{xy}^{(k)}\nu_{yx}^{(k)}} \qquad (17)$$

where $G_{xy}^{(k)}$ is axial shear modulus of the timber and γ_{xy} is in-plane shear strain. The phase average stress terms are possible because they integrate over volume of a single phase with constant properties. The constant properties can be removed from the integral. For example:

$$\begin{aligned}
\overline{\sigma_{xx}^{(1)}} &= \frac{1}{v_1}\int_{v_1}\left[Q_{xx}^{(1)}\left(\varepsilon_{xx} - \alpha_{xx}^{(1)}\Delta T\right) + Q_{xy}^{(1)}\left(\varepsilon_{yy} - \alpha_{yy}^{(1)}\Delta T\right)\right]dV \\
&= Q_{xx}^{(1)}\left(\frac{1}{v_1}\int_{v_1}\varepsilon_{xx}\,dV - \alpha_{xx}^{(1)}\Delta T\right) + Q_{xy}^{(1)}\left(\frac{1}{v_1}\int_{v_1}\varepsilon_{yy}\,dV - \alpha_{yy}^{(1)}\Delta T\right) \\
&= Q_{xx}^{(1)}(\overline{\varepsilon_{xx}^{(1)}} - \alpha_{xx}^{(1)}\Delta T) + Q_{xy}^{(1)}(\overline{\varepsilon_{yy}^{(1)}} - \alpha_{yy}^{(1)}\Delta T)
\end{aligned} \qquad (18)$$

These six equations have eight unknowns (phase average stresses and strains), but for an orthogonally-cracked CLT unit cell under mechanical loads only ($\Delta T = 0$), the two average strains in uncracked layer directions must equal the global panel strains due to mechanical loading by only σ_{10} or:

$$\overline{\varepsilon_{xx}^{(2m)}} = \varepsilon_{10}^{(m)} = \frac{\sigma_{10}}{E_{11}(a,b)} \qquad \text{and} \qquad \overline{\varepsilon_{yy}^{(1m)}} = \varepsilon_{20}^{(m)} = -\frac{\nu_{12}\sigma_{10}}{E_{11}(a,b)} \qquad (19)$$

Solving for the remaining six unknowns, and replacing (1) and (2) with (1m) and (2m) to indicate mechanical stresses, the needed phase average mechanical stresses are:

$$\overline{\sigma_{xx}^{(1m)}} = \frac{Q_{xx}^{(1)}Q_{yy}^{(2)}E_{11}(a,b) - Q_{xx}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2 - Q_{xy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)\nu_{12}(a,b)V_1}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{11}(a,b)V_1}\sigma_{10} \qquad (20)$$

$$\overline{\sigma_{yy}^{(1m)}} = \frac{Q_{xx}^{(1)}Q_{yy}^{(2)}E_{11}(a,b) - Q_{xy}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2 - Q_{yy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)\nu_{12}(a,b)V_1}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{11}(a,b)V_1}\sigma_{10} \qquad (21)$$

$$\overline{\sigma_{xx}^{(2m)}} = \frac{-Q_{xy}^{(1)}Q_{xy}^{(2)}E_{11}(a,b) + Q_{xx}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2 + Q_{xy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)\nu_{12}(a,b)V_1}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{11}(a,b)V_2}\sigma_{10} \qquad (22)$$

$$\overline{\sigma_{yy}^{(2m)}} = \frac{-Q_{xy}^{(1)}Q_{yy}^{(2)}E_{11}(a,b) + Q_{xy}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2 + Q_{yy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)\nu_{12}(a,b)V_1}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{11}(a,b)V_2}\sigma_{10} \qquad (23)$$

Substituting these phase average stresses into Eq. (15) and simplifying leads to:

$$\alpha_{11}(a,b)\Delta T = \Delta_0 - \frac{V_2\Delta_2}{E_{11}(a,b)} + \frac{V_1\Delta_1\nu_{12}(a,b)}{E_{11}(a,b)} \tag{24}$$

where

$$\Delta_0 = \left(\alpha_{xx}^{(2)} + \frac{Q_{xx}^{(1)}(\alpha_{xx}^{(1)}-\alpha_{xx}^{(2)}) + Q_{xy}^{(1)}(\alpha_{yy}^{(1)}-\alpha_{yy}^{(2)})}{Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)}} Q_{yy}^{(2)}\right)\Delta T \tag{25}$$

$$\Delta_1 = -\left(Q_{xy}^{(2)}(\alpha_{xx}^{(1)}-\alpha_{xx}^{(2)}) + Q_{yy}^{(2)}(\alpha_{yy}^{(1)}-\alpha_{yy}^{(2)})\right)\Delta T\,\frac{Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2}{Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)}} \tag{26}$$

$$\Delta_2 = \left(Q_{xx}^{(1)}(\alpha_{xx}^{(1)}-\alpha_{xx}^{(2)}) + Q_{xy}^{(1)}(\alpha_{yy}^{(1)}-\alpha_{yy}^{(2)})\right)\Delta T\,\frac{Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2}{Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)}} \tag{27}$$

To find panel CTE in the 2 direction, consider mechanical loading in direction 2 only. The Levin (1967) equation now becomes

$$\sigma_{20}\alpha_{22}(a,b)\Delta T = \sum_{k=1}^{2} V_k\left(\overline{\sigma_{xx}^{(km)}}\alpha_{xx}^{(k)} + \overline{\sigma_{yy}^{(km)}}\alpha_{yy}^{(k)}\right) \tag{28}$$

The first two equations in Eq. (16) change to:

$$\begin{aligned} 0 &= V_1\overline{\sigma_{xx}^{(1m)}} + V_2\overline{\sigma_{xx}^{(2m)}} \\ \sigma_{20} &= V_1\overline{\sigma_{yy}^{(1m)}} + V_2\overline{\sigma_{yy}^{(2m)}} \end{aligned}$$

and the two known phase average strains due to mechanical loading (now by only σ_{20}) change to:

$$\overline{\varepsilon_{xx}^{(2m)}} = \varepsilon_{10}^{(m)} = -\frac{\nu_{21}(a,b)\sigma_{20}}{E_{22}(a,b)} \qquad \text{and} \qquad \overline{\varepsilon_{yy}^{(1m)}} = \varepsilon_{20}^{(m)} = \frac{\sigma_{20}}{E_{11}(a,b)} \tag{29}$$

Solving for the remaining six unknowns, the needed phase average mechanical stresses are:

$$\overline{\sigma_{xx}^{(1m)}} = \frac{-Q_{xx}^{(1)}Q_{xy}^{(2)}E_{22}(a,b) + Q_{xy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 + Q_{xx}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)\nu_{21}(a,b)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{22}(a,b)V_1}\sigma_{20} \tag{30}$$

$$\overline{\sigma_{yy}^{(1m)}} = \frac{-Q_{xy}^{(1)}Q_{xy}^{(2)}E_{22}(a,b) + Q_{yy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 + Q_{xy}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)\nu_{21}(a,b)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{22}(a,b)V_1}\sigma_{20} \tag{31}$$

$$\overline{\sigma_{xx}^{(2m)}} = \frac{Q_{xx}^{(1)}Q_{xy}^{(2)}E_{22}(a,b) - Q_{xy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 - Q_{xx}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)\nu_{21}(a,b)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{22}(a,b)V_2}\sigma_{20} \tag{32}$$

$$\overline{\sigma_{yy}^{(2m)}} = \frac{Q_{xx}^{(1)}Q_{yy}^{(2)}E_{22}(a,b) - Q_{yy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 - Q_{xy}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)\nu_{21}(a,b)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})E_{22}(a,b)V_2}\sigma_{20} \tag{33}$$

Substituting these phase average stresses into Eq. (28) and simplifying leads to:

$$\alpha_{22}(a,b)\Delta T = \Delta_0' - \frac{V_1\Delta_1}{E_{22}(a,b)} + \frac{V_2\Delta_2\nu_{12}(a,b)}{E_{11}(a,b)} \tag{34}$$

where

$$\Delta_0' = \left(\alpha_{yy}^{(1)} + \frac{Q_{xy}^{(2)}(\alpha_{xx}^{(1)}-\alpha_{xx}^{(2)}) + Q_{yy}^{(2)}(\alpha_{yy}^{(1)}-\alpha_{yy}^{(2)})}{Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)}} Q_{xx}^{(1)}\right)\Delta T \tag{35}$$

The final term, $\langle\sigma^r \cdot \alpha\Delta T\rangle$, cannot use the Levin (1967) equation because residual stresses do not correspond to a state of applied mechanical stress. We can, however, directly evaluate it by substituting phase averaged residual stresses into Eq. (6). The phase average residual stresses can be found by solving Eq. (16) when $\sigma_{10} = 0$ and $\Delta T \neq 0$

and replacing (1) and (2) with $(1r)$ and $(2r)$ to indicate residual stresses. The two known phase average strains due to thermal loading only are:

$$\overline{\varepsilon_{xx}^{(2r)}} = \alpha_{11}(a,b)\Delta T \qquad \text{and} \qquad \overline{\varepsilon_{yy}^{(1r)}} = \alpha_{22}(a,b)\Delta T \tag{36}$$

Solving for the remaining six unknowns, the needed phase average residual stresses are:

$$\overline{\sigma_{xx}^{(1r)}} = \frac{-\big(\alpha_{yy}^{(1)} - \alpha_{22}(a,b)\big)Q_{xy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 + \big(\alpha_{xx}^{(2)} - \alpha_{11}(a,b)\big)Q_{xx}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})V_1}\Delta T \tag{37}$$

$$\overline{\sigma_{yy}^{(1r)}} = \frac{-\big(\alpha_{yy}^{(1)} - \alpha_{22}(a,b)\big)Q_{yy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 + \big(\alpha_{xx}^{(2)} - \alpha_{11}(a,b)\big)Q_{xy}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})V_1}\Delta T \tag{38}$$

$$\overline{\sigma_{xx}^{(2r)}} = \frac{\big(\alpha_{yy}^{(1)} - \alpha_{22}(a,b)\big)Q_{xy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 - \big(\alpha_{xx}^{(2)} - \alpha_{11}(a,b)\big)Q_{xx}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})V_2}\Delta T \tag{39}$$

$$\overline{\sigma_{yy}^{(2r)}} = \frac{\big(\alpha_{yy}^{(1)} - \alpha_{22}(a,b)\big)Q_{yy}^{(2)}(Q_{xx}^{(1)}Q_{yy}^{(1)} - {Q_{xy}^{(1)}}^2)V_1 - \big(\alpha_{xx}^{(2)} - \alpha_{11}(a,b)\big)Q_{xy}^{(1)}(Q_{xx}^{(2)}Q_{yy}^{(2)} - {Q_{xy}^{(2)}}^2)V_2}{(Q_{xx}^{(1)}Q_{yy}^{(2)} - Q_{xy}^{(1)}Q_{xy}^{(2)})V_2}\Delta T \tag{40}$$

Substituting these phase average residual stresses into Eq. (6) and simplifying results in

$$\langle \sigma^r \cdot \alpha \Delta T \rangle = V_1\Delta_1\big(\alpha_{yy}^{(1)} - \alpha_{22}(a,b)\big)\Delta T + V_2\Delta_2\big(\alpha_{xx}^{(2)} - \alpha_{11}(a,b)\big)\Delta T \tag{41}$$

Substituting Eqs. (12), (14) and (41) into Eq. (1) gives:

$$G = \frac{V}{2}\frac{d}{dA}\left[\frac{\sigma_{10}^2}{E_{11}(a,b)} - \frac{2\nu_{12}(a,b)\sigma_{10}\sigma_{20}}{E_{11}(a,b)} + \frac{\sigma_{20}^2}{E_{22}(a,b)} + 2\sigma_{10}\alpha_{11}(a,b)\Delta T + 2\sigma_{20}\alpha_{22}(a,b)\Delta T \right.$$
$$\left. -V_1\Delta_1\alpha_{22}(a,b)\Delta T - V_2\Delta_2\alpha_{11}(a,b)\Delta T\right] \tag{42}$$

where derivatives of constant $\alpha_{yy}^{(1)}$ and $\alpha_{xx}^{(2)}$ terms in Eq. (41) with respect to fracture area, dA, are zero because those are constant phase properties. Eliminating effective panel CTEs using Eqs. (24) and (34) (and using $d\Delta_0/dA = d\Delta_0'/dA$=0) gives:

$$\begin{aligned}
G = {} & \frac{V}{2}\frac{d}{dA}\left[\frac{\sigma_{10}^2}{E_{11}(a,b)} - \frac{2\nu_{12}(a,b)\sigma_{10}\sigma_{20}}{E_{11}(a,b)} + \frac{\sigma_{20}^2}{E_{22}(a,b)} - \frac{2\sigma_{10}V_2\Delta_2}{E_{11}(a,b)} + \frac{2\sigma_{10}V_1\Delta_1\nu_{12}(a,b)}{E_{11}(a,b)} - \frac{2\sigma_{20}V_1\Delta_1}{E_{22}(a,b)}\right. \\
& \left. + \frac{2\sigma_{20}V_2\Delta_2\nu_{12}(a,b)}{E_{11}(a,b)} + \frac{V_1^2\Delta_1^2}{E_{22}(a,b)} - \frac{V_1\Delta_1V_2\Delta_2\nu_{12}(a,b)}{E_{11}(a,b)} + \frac{V_2^2\Delta_2^2}{E_{11}(a,b)} - \frac{V_2\Delta_2V_1\Delta_1\nu_{12}(a,b)}{E_{11}(a,b)}\right] \\
= {} & \frac{V}{2}\frac{d}{dA}\left[\frac{\sigma_{10}^2 - 2\sigma_{10}V_2\Delta_2 + V_2^2\Delta_2^2}{E_{11}(a,b)} - \frac{2\nu_{12}(a,b)\big(\sigma_{10}\sigma_{20} - \sigma_{10}V_1\Delta_1 - \sigma_{20}V_2\Delta_2 + V_1\Delta_1V_2\Delta_2\big)}{E_{11}(a,b)}\right. \\
& \left. + \frac{\sigma_{20}^2 - 2\sigma_{20}V_1\Delta_1 + V_1^2\Delta_1^2}{E_{22}(a,b)}\right] \\
= {} & \frac{V}{2}\frac{d}{dA}\left[{\sigma_1^*}^2\frac{1}{E_{11}(a,b)} - 2\sigma_1^*\sigma_2^*\frac{\nu_{12}(a,b)}{E_{11}(a,b)} + {\sigma_2^*}^2\frac{1}{E_{22}(a,b)}\right]
\end{aligned} \tag{43}$$

where σ_i^* are effective stresses:

$$\sigma_1^* = \sigma_{10} - V_2\Delta_2 \qquad \text{and} \qquad \sigma_2^* = \sigma_{20} - V_1\Delta_1 \tag{44}$$

Equation (43) is an exact expression for energy release rate for cracking in a CLT panel under combined in-plane mechanical loads and residual thermal stresses.

One more simplification is possible. For cracking only in layer 1 where a changes but b remains constant, the total G must scale with a *sum* of mechanical and residual stresses (Nairn, 1997). As a consequence, the key terms:

$$\sigma_1^{*2}\left(\frac{d}{dA}\frac{1}{E_{11}(a,b)}\right)_b - 2\sigma_1^*\sigma_2^*\left(\frac{d}{dA}\frac{\nu_{12}(a,b)}{E_{11}(a,b)}\right)_b + \sigma_2^{*2}\left(\frac{d}{dA}\frac{1}{E_{22}(a,b)}\right)_b \tag{45}$$

must be complete square in effective stresses and therefore it must be that:

$$\left(\frac{d}{dA}\frac{\nu_{12}(a,b)}{E_{11}(a,b)}\right)_b = \sqrt{\left(\frac{d}{dA}\frac{1}{E_{11}(a,b)}\right)_b\left(\frac{d}{dA}\frac{1}{E_{22}(a,b)}\right)_b} \tag{46}$$

Expressed another way (as done by McCartney (1993) who derived if by different methods), an exact relation between derivatives of effective panel properties is:

$$\frac{\left(\frac{d}{dA}\frac{\nu_{12}(a,b)}{E_{11}(a,b)}\right)_b}{\left(\frac{d}{dA}\frac{1}{E_{11}(a,b)}\right)_b} = \frac{\left(\frac{d}{dA}\frac{1}{E_{22}(a,b)}\right)_b}{\left(\frac{d}{dA}\frac{\nu_{12}(a,b)}{E_{11}(a,b)}\right)_b} \tag{47}$$

An analogous expression holds for cracking in only layer 2 where b changes but a remains constant. These results were confirmed by stress modeling as a function of crack density using results in Nairn (2017). Substitution of this exact relation into Eq. (43) leads to

$$G = \frac{V\sigma_0^{*2}}{2}\frac{d}{dA}\left[\frac{\xi}{\sqrt{E_{11}(a,b)}} + \frac{1-\xi}{\sqrt{E_{22}(a,b)}}\right]^2 \tag{48}$$

where $\sigma_0^* = \sigma_1^* + \sigma_2^*$ and $\xi = \sigma_1^*/\sigma_0^*$. Because this result uses the relation that assumes cracking in one layer only, it only holds for changes in a or b while the other one remains constant. It can be used to predict cracking in either layer by considering a sequence of alternating layer cracking events.

Although this above analysis considered only residual thermal stresses, the results are trivially extended to combine thermal and moisture residual stresses by replacing all instances of $\alpha_{ii}^{(k)}\Delta T$ with $\alpha_{ii}^{(k)}\Delta T + \beta_{ii}^{(k)}\Delta c$. These terms only enter the Δ_i terms.

2 Uncracked Laminate Properties

Several calculations in the paper depend on uncracked laminate properties — E_{11}^0, E_{22}^0, ν_{12}^0, ν_{21}^0, α_{11}^0, and α_{22}^0. If the hypothetical, uncracked CLT panel is consider to be a laminate of homogeneous layers, these properties are easily calculated with a special case of laminated plate theory (Christenson, 1979) for in-plane loading of a laminate consisting only of 0° and 90° plies. The force (N_{ij}) and resultants for a symmetric n-layer composite are given by:

$$\begin{pmatrix} N_{11} \\ N_{22} \\ N_{12} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \gamma_{12}^0 \end{pmatrix} \tag{49}$$

where ε_{ij}^0 are in-plane strains applied to the laminate, and

$$A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} t^{(k)} \quad \text{and} \tag{50}$$

where $Q_{ij}^{(k)}$ are elements of the plane-stress stiffness matrix for layer k. For a CLT panel, the only non-zero elements of the **A** matrix are:

$$A_{11} = 2h(V_1Q_{xx}^{(1)} + V_2Q_{xx}^{(2)}) = 2h\langle Q_{xx}\rangle \tag{51}$$

$$A_{12} = 2h(V_1Q_{xy}^{(1)} + V_2Q_{xy}^{(2)}) = 2h\langle Q_{xy}\rangle \tag{52}$$

$$A_{22} = 2h(V_1Q_{yy}^{(1)} + V_2Q_{yy}^{(2)}) = 2h\langle Q_{yy}\rangle \tag{53}$$

$$A_{66} = 2h(V_1G_{xy}^{(1)} + V_2G_{xy}^{(2)}) = 2h\langle G_{xx}\rangle \tag{54}$$

where angle brackets average that property over all layers. The mechanical properties of the laminate can be derived from:

$$E_{11}^0 = \frac{1}{2ha_{11}}, \quad E_{11}^0 = \frac{1}{2ha_{22}}, \quad \nu_{12}^0 = -\frac{a_{21}}{a_{11}}, \quad \text{and} \quad \nu_{12}^0 = -\frac{a_{21}}{a_{22}} \tag{55}$$

where $\mathbf{a} = \mathbf{A}^{-1}$. In terms of elements of the $\mathbf{A}$ matrix, these properties are:

$$E_{11}^0 = \frac{A_{11}A_{22} - A_{12}^2}{2hA_{22}}, \quad E_{11}^0 = \frac{A_{11}A_{22} - A_{12}^2}{2hA_{11}}, \quad \nu_{12}^0 = \frac{A_{12}}{A_{22}}, \quad \text{and} \quad \nu_{12}^0 = \frac{A_{12}}{A_{11}} \tag{56}$$

Substituting the above results for A_{ij} gives:

$$E_{11}^0 = \langle Q_{xx}\rangle(1-\nu_{21}^0\nu_{12}^0),\ E_{22}^0 = \langle Q_{yy}\rangle(1-\nu_{21}^0\nu_{12}^0),\ \nu_{12}^0 = \frac{\langle Q_{xy}\rangle}{\langle Q_{yy}\rangle}, \text{ and } \nu_{21}^0 = \frac{\langle Q_{xy}\rangle}{\langle Q_{xx}\rangle} \tag{57}$$

Laminated plate theory can also be used to find α_{11}^0 and α_{22}^0, but the results are the same as $\alpha_{11}(a,b)$ and $\alpha_{22}(a,b)$ in the paper by replacing $E_{11}(a,b)$, $E_{22}(a,b)$ and $\nu_{12}(a,b)$ with E_{11}^0, E_{22}^0, and ν_{12}^0.

3 Generalized and Simplified Analysis for Tensile Moduli

The analysis for tensile moduli in a CLT panel with possibly different layers is done by finding the complementary energy in the panel based on an assumed stress state. The assumed stress state is posed in terms of two unknown functions that are found by minimizing the complementary energy. Once the functions are found, they are substituted back into the complementary energy to calculate a lower bound on the two in-plane tensile modulus, $E_{11}(a,b)$ and $E_{22}(a,b)$. The background details are found in Hashin (1987) and Nairn (2017). This sections repeats those analyses to work when the surface and core layers are different. This section also shows how the simplified expressions given in the paper are derived.

The analysis begins with the following 3D "perturbation" stress state for layers 1 and 2 (Hashin, 1987; Nairn, 2017) when loading in direction $i = 1$ or 2. Using the nomenclature from Nairn (2017), those stresses are:

$$\begin{array}{llll} \sigma_{xx}^{(1)} = -\sigma_0 k_{xi}\phi_i(x) & \sigma_{xx}^{(2)} = \frac{1}{\lambda_i}\sigma_0 k_{xi}\phi_i(x) & \sigma_{xy}^{(1)} = 0 & \sigma_{xy}^{(2)} = 0 \\ \sigma_{yy}^{(1)} = -\sigma_0 k_{yi}\psi_i(y) & \sigma_{yy}^{(2)} = \frac{1}{\lambda_i}\sigma_0 k_{yi}\psi_i(y) & \sigma_{xz}^{(1)} = \sigma_0 k_{xi}\phi_i'(x)z & \sigma_{xz}^{(2)} = \frac{1}{\lambda_i}\sigma_0 k_{xi}\phi_i'(x)(h-z) \end{array} \tag{58}$$

$$\sigma_{yz}^{(1)} = \sigma_0 k_{yi}\psi_i'(y)z \qquad \sigma_{zz}^{(1)} = \begin{cases} \sigma_0\left[k_{xi}\phi_1''(x) + k_{yi}\psi_1''(y)\right]\frac{1}{2}(ht_1 - z^2) & \text{direction 1} \\ -\sigma_0\left[k_{xi}\phi_2''(x) + k_{yi}\psi_2''(y)\right]\frac{1}{2}z^2 & \text{direction 2} \end{cases} \tag{59}$$

$$\sigma_{yz}^{(2)} = \frac{1}{\lambda_i}\sigma_0 k_{yi}\psi_i'(y)(h-z) \qquad \sigma_{zz}^{(2)} = \begin{cases} \sigma_0\left[k_{xi}\phi_1''(x) + k_{yi}\psi_1''(y)\right]\frac{1}{2\lambda_i}(h-z)^2 & \text{direction 1} \\ \sigma_0\left[k_{xi}\phi_2''(x) + k_{yi}\psi_2''(y)\right]\frac{1}{2\lambda_i}\left(h(t_2-2z)+z^2\right) & \text{direction 2} \end{cases} \tag{60}$$

where $\phi_i(x)$ and $\psi_i(y)$ are four unknown functions (two for each loading direction), $\lambda_1 = t_2/t_1$, and $\lambda_2 = t_1/t_2$. The stiffnesses, k_{xi} and k_{yi}, give the stress in layer 1 of the uncracked laminate due to uniaxial load in direction i and are easily calculated from laminated plate theory. The only non-zero initial stresses needed in the analysis are:

$$\sigma_{xx}^{(1,0)} = k_{xi}\sigma_0 \qquad \text{and} \qquad \sigma_{yy}^{(1,0)} = k_{yi}\sigma_0 \tag{61}$$

The term "perturbation" stresses means the *change* in stresses between a hypothetical panel with no cracks and a panel with crack spacings a and b in the core and surface layers, respectively. The boundary conditions for $\phi_i(x)$ and $\psi_i(y)$ assume the crack surfaces are stress free. Note that by exploiting an assumption that all layers are the same, Nairn (2017) derived the loading direction 2 result by interchanging the meaning of layers 1 and 2. In other words, for direction 2, layer 1 is surface layer and layer 2 is core layer. This approach does not work when the layers have different properties, but accounting for different properties is easily done by generalizing the complementary energy calculation. Also note that k_{xi} and k_{yi} are referred to in the current paper as "influence coefficients" for translation of applied mechanical stress into layer stresses in the uncracked panel (these coefficients are defined in the paper). Note that compared to Nairn (2017), the influence coefficients defined in the current paper are $k_{x1}^{(1)} = k_{x1}$ and $k_{y1}^{(1)} = k_{y1}$ but because Nairn (2017) rotated x and y for loading in direction 2, $k_{x2}^{(2)} = k_{y2}$ and $k_{y2}^{(2)} = k_{x2}$.

We need to find U_{ci} or the complementary energy calculated from perturbation stresses alone when loading only in direction i, which is given by:

$$U_{ci} = \frac{1}{2}\int \sigma \cdot \mathsf{S}\sigma\, dV = t_i^2 \int_{-\rho_i}^{\rho_i}\int_{-\chi_i}^{\chi_i}\left(\int_0^{t_1} W^{(1)}\, dz + \int_{t_1}^{h} W^{(2)}\, dz\right) d\eta\, d\xi \tag{62}$$

where $\xi = x/t_i$ and $\eta = y/t_i$ are dimensionless coordinates and $\rho_1 = a/t_1$, $\rho_2 = b/t_2$, $\chi_1 = b/t_1$, and $\chi_2 = a/t_2$ are dimensionless crack spacings. This equation is again using the notation from Nairn (2017), but note that the direction 2 analysis in that paper rotated meaning of x and y axes. The terms $W^{(1)}$ and $W^{(2)}$ are complementary energy in each layer. Generalizing that calculation for different surface or core layers when loading in the 1 direction:

$$2W^{(1)} = \frac{{\sigma_{xx}^{(1)}}^2}{E_{xx}^{(1)}} + \frac{{\sigma_{yy}^{(1)}}^2}{E_{yy}^{(1)}} + \frac{{\sigma_{zz}^{(1)}}^2}{E_{zz}^{(1)}} - \frac{2\sigma_{yy}^{(1)}(\nu_{yx}^{(1)}\sigma_{xx}^{(1)} + \nu_{yz}^{(1)}\sigma_{zz}^{(1)})}{E_{yy}^{(1)}} - \frac{2\nu_{xz}^{(1)}\sigma_{xx}^{(1)}\sigma_{zz}^{(1)}}{E_{xx}^{(1)}} + \frac{{\sigma_{yz}^{(1)}}^2}{G_{yz}^{(1)}} + \frac{{\sigma_{xz}^{(1)}}^2}{G_{xz}^{(1)}} \tag{63}$$

$$2W^{(2)} = \frac{{\sigma_{xx}^{(2)}}^2}{E_{xx}^{(2)}} + \frac{{\sigma_{yy}^{(2)}}^2}{E_{yy}^{(2)}} + \frac{{\sigma_{zz}^{(2)}}^2}{E_{zz}^{(2)}} - \frac{2\sigma_{yy}^{(2)}(\nu_{yx}^{(2)}\sigma_{xx}^{(2)} + \nu_{yz}^{(2)}\sigma_{zz}^{(2)})}{E_{yy}^{(2)}} - \frac{2\nu_{xz}^{(2)}\sigma_{xx}^{(2)}\sigma_{zz}^{(2)}}{E_{xx}^{(2)}} + \frac{{\sigma_{yz}^{(2)}}^2}{G_{yz}^{(2)}} + \frac{{\sigma_{xz}^{(2)}}^2}{G_{xz}^{(2)}} \tag{64}$$

where E, G, and ν are anisotropic mechanical properties of core (superscript 1) and surface (superscript 2) layers. The subscripts x and y on mechanical properties refer to x and y axes in Fig. 1. For loading in direction 2, $W^{(1)}$ and $W^{(2)}$ refer to complementary energy in surface and core layers (*i.e.*, meaning of 1 and 2 have been interchanged). Recognizing the change in notation, the complementary energies in the surface layer (now $W^{(1)}$) and the core layer (now $W^{(2)}$) that generalize Nairn (2017) for different layer properties are:

$$2W^{(1)} = \frac{{\sigma_{xx}^{(1)}}^2}{E_{yy}^{(2)}} + \frac{{\sigma_{yy}^{(1)}}^2}{E_{xx}^{(2)}} + \frac{{\sigma_{zz}^{(1)}}^2}{E_{zz}^{(2)}} - \frac{2\sigma_{yy}^{(1)}(\nu_{xy}^{(2)}\sigma_{xx}^{(1)} + \nu_{xz}^{(2)}\sigma_{zz}^{(1)})}{E_{xx}^{(2)}} - \frac{2\nu_{yz}^{(2)}\sigma_{xx}^{(1)}\sigma_{zz}^{(1)}}{E_{yy}^{(2)}} + \frac{{\sigma_{yz}^{(1)}}^2}{G_{xz}^{(2)}} + \frac{{\sigma_{xz}^{(1)}}^2}{G_{yz}^{(2)}} \tag{65}$$

$$2W^{(2)} = \frac{{\sigma_{xx}^{(2)}}^2}{E_{yy}^{(1)}} + \frac{{\sigma_{yy}^{(2)}}^2}{E_{xx}^{(1)}} + \frac{{\sigma_{zz}^{(2)}}^2}{E_{zz}^{(1)}} - \frac{2\sigma_{yy}^{(2)}(\nu_{xy}^{(1)}\sigma_{xx}^{(2)} + \nu_{xz}^{(1)}\sigma_{zz}^{(2)})}{E_{xx}^{(1)}} - \frac{2\nu_{yz}^{(1)}\sigma_{xx}^{(2)}\sigma_{zz}^{(2)}}{E_{yy}^{(1)}} + \frac{{\sigma_{yz}^{(2)}}^2}{G_{xz}^{(1)}} + \frac{{\sigma_{xz}^{(2)}}^2}{G_{yz}^{(1)}} \tag{66}$$

Note that superscripts on stresses refer to stress state in Eq. (60) (*i.e.*, using nomenclature from Nairn (2017)). In contrast, superscripts on mechanical properties are now using 1 and 2 for core and surface layers, which may differ. Also note that layer properties are accounting for interchange of x and y axes done in the Nairn (2017) analysis for loading direction 2. The subscripts x and y here are mechanical properties along x and y axes in Fig. 1.

Substituting $W^{(1)}$ and $W^{(2)}$ for direction $i = 1$ loading into Eq. (62) and much (easy) algebra leads to

$$\begin{aligned} 2U_{c1} = \sigma_0^2 t_1^3 \int_{-\rho_1}^{\rho_1}\int_{-\chi_1}^{\chi_1} \big(& A_0 {k_{x1}}^2\phi^2 + 2B_0 k_{x1}k_{y1}\phi\psi + C_0 {k_{y1}}^2\psi^2 + A_1 {k_{x1}}^2\phi'^2 + B_1 {k_{y1}}^2\psi'^2 \\ & + A_2 k_{x1}\phi(k_{x1}\phi'' + k_{y1}\psi'') + B_2 k_{y1}\psi(k_{x1}\phi'' + k_{y1}\psi'') + C_2(k_{x1}\phi'' + k_{y1}\psi'')^2\big) d\xi\, d\eta \end{aligned} \tag{67}$$

where $\phi = \phi_1(\xi)$ and $\psi = \psi_1(\eta)$ and the constants are

$$\begin{aligned} A_0 &= \tfrac{1}{E_{xx}^{(1)}} + \tfrac{1}{\lambda E_{xx}^{(2)}} & B_0 &= -\left(\tfrac{\nu_{yx}^{(1)}}{E_{yy}^{(1)}} + \tfrac{\nu_{yx}^{(2)}}{\lambda E_{yy}^{(2)}}\right) & C_0 &= \tfrac{1}{E_{yy}^{(1)}} + \tfrac{1}{\lambda E_{yy}^{(2)}} \\ A_1 &= \tfrac{1}{3G_{xz}^{(1)}} + \tfrac{\lambda}{3G_{xz}^{(2)}} & B_1 &= \tfrac{1}{3G_{yz}^{(1)}} + \tfrac{\lambda}{3G_{yz}^{(2)}} & A_2 &= \tfrac{(3\lambda+2)\nu_{xz}^{(1)}}{3E_{xx}^{(1)}} - \tfrac{\lambda\nu_{xz}^{(2)}}{3E_{xx}^{(2)}} \\ B_2 &= \tfrac{(3\lambda+2)\nu_{yz}^{(1)}}{3E_{yy}^{(1)}} - \tfrac{\lambda\nu_{yz}^{(2)}}{3E_{yy}^{(2)}} & C_2 &= \tfrac{1}{60}\left(\tfrac{8+20\lambda+15\lambda^2}{E_{zz}^{(1)}} + \tfrac{3\lambda^3}{E_{zz}^{(2)}}\right) \end{aligned} \tag{68}$$

and $\lambda = \lambda_1 = t_2/t_1$

Substituting $W^{(1)}$ and $W^{(2)}$ for direction $i = 2$ loading into Eq. (62) and much (easy) algebra leads to

$$\begin{aligned} 2U_{c1} = \sigma_0^2 t_2^3 \int_{-\rho_2}^{\rho_2}\int_{-\chi_2}^{\chi_2} \big(& A_0' {k_{x2}}^2\phi^2 + 2B_0' k_{x2}k_{y2}\phi\psi + C_0' {k_{y2}}^2\psi^2 + A_1' {k_{x2}}^2\phi'^2 + B_1' {k_{y2}}^2\psi'^2 \\ & + A_2' k_{x2}\phi(k_{x2}\phi'' + k_{y1}\psi'') + B_2' k_{y2}\psi(k_{x2}\phi'' + k_{y2}\psi'') + C_2'(k_{x2}\phi'' + k_{y2}\psi'')^2\big) d\xi\, d\eta \end{aligned} \tag{69}$$

where $\phi = \phi_2(\xi)$ and $\psi = \psi_2(\eta)$ and the constants are

$$\begin{array}{lll} A'_0 = \frac{1}{E_{yy}^{(2)}} + \frac{1}{\lambda_2 E_{yy}^{(1)}} & B'_0 = -\left(\frac{\nu_{xy}^{(2)}}{E_{xx}^{(2)}} + \frac{\nu_{xy}^{(1)}}{\lambda_2 E_{xx}^{(1)}}\right) & C'_0 = \frac{1}{E_{xx}^{(2)}} + \frac{1}{\lambda_2 E_{xx}^{(1)}} \\ A'_1 = \frac{1}{3G_{yz}^{(2)}} + \frac{\lambda_2}{3G_{yz}^{(1)}} & B'_1 = \frac{1}{3G_{xz}^{(2)}} + \frac{\lambda_2}{3G_{xz}^{(1)}} & A'_2 = \frac{(3+2\lambda_2)\nu_{yz}^{(1)}}{3E_{yy}^{(1)}} - \frac{\nu_{yz}^{(2)}}{3E_{yy}^{(2)}} \\ B'_2 = \frac{(3+2\lambda_2)\nu_{xz}^{(1)}}{3E_{xx}^{(1)}} - \frac{\nu_{xz}^{(2)}}{3E_{xx}^{(2)}} & C'_2 = \frac{1}{60}\left(\frac{8\lambda_2^2+20\lambda_2+15}{E_{zz}^{(1)}} + \frac{3}{E_{zz}^{(2)}}\right) & \end{array} \tag{70}$$

Noting that $\lambda_2 = 1/\lambda$, all primed constants for loading in direction 2 can be expressed in terms of direction 1 constants using:

$$A'_0 = \lambda C_0,\ B'_0 = \lambda B_0,\ C'_0 = \lambda A_0,\ A'_1 = \frac{B_1}{\lambda}, B'_1 = \frac{A_1}{\lambda},\ A'_2 = \frac{B_2}{\lambda},\ B'_2 = \frac{A_2}{\lambda}.\ C'_2 = \frac{C_2}{\lambda^3} \tag{71}$$

The next steps are to minimize complementary energy in each direction to solve for unknown functions $\phi_1(\xi)$, $\psi_1(\eta)$, $\phi_2(\xi)$, and $\psi_2(\eta)$ and then substitute them back into complementary energy to find modulus. Because the complementary energies are expressed in terms of constants (A_i, B_i, and C_i), the solution methods given in Hashin (1987) and Nairn (2017) are the same and need not be repeated. In other words, the only change needed to account for different core and surface layer properties is to use a different set of constants. The solution process is described in the body and appendix of Hashin (1987). The *Online Resource* for Nairn (2017) extends the solution to loading in direction 2 (note: that *Online Resource* has a few typos in preliminary equations, but final expressions for the solutions are correct). The new form given in the "Explicit Calculations" section of the current paper coordinates all those solutions into a single function that resulted from minimizing complementary energies. Thus the current paper includes sufficient equations to do all calculations presented in the paper.

4 Drying of Surface Layers Compared to Dying All Layers

Imagine a CLT panel in which the moisture content of layer k is Δc_k relative to the equilibrium moisture content of the panel after fabricated and left to reach equilibrium moisture content. The panel dimensions will change from initial bonding to equilibrium. For sample calculations, it suffices to consider a panel with no cracks. After reaching equilibrium, the global panel strains will change to ε_{11} and ε_{22} and will be same in all layers. The global relations for this problem are:

$$\begin{aligned} 0 &= V_1\overline{\sigma_{xx}^{(1r)}} + V_2\overline{\sigma_{xx}^{(2r)}} \\ 0 &= V_1\overline{\sigma_{yy}^{(1)}} + V_2\overline{\sigma_{yy}^{(2)}} \\ \overline{\sigma_{xx}^{(1r)}} &= Q_{xx}^{(1)}(\varepsilon_{11} - \beta_{xx}^{(1)}\Delta c_1) + Q_{xy}^{(1)}(\varepsilon_{22} - \beta_{yy}^{(1)}\Delta c_1) \\ \overline{\sigma_{yy}^{(1)}} &= Q_{xy}^{(1)}(\varepsilon_{11} - \beta_{xx}^{(1)}\Delta c_1) + Q_{yy}^{(1)}(\varepsilon_{22} - \beta_{yy}^{(1)}\Delta c_1) \\ \overline{\sigma_{xx}^{(2)}} &= Q_{xx}^{(2)}(\varepsilon_{11} - \beta_{xx}^{(2)}\Delta c_2) + Q_{xy}^{(2)}(\varepsilon_{22} - \beta_{yy}^{(2)}\Delta c_2) \\ \overline{\sigma_{yy}^{(2)}} &= Q_{xy}^{(2)}(\varepsilon_{11} - \beta_{xx}^{(2)}\Delta c_2) + Q_{yy}^{(1)}(\varepsilon_{22} - \beta_{yy}^{(2)}\Delta c_2) \end{aligned} \tag{72}$$

This analysis assumes only moisture induced residual stresses ($\Delta T = 0$) and considers layers drier than panel equilibrium, $\Delta c_k > 0$. Solving these six equation can calculate the transverse stresses in layers 1 and 2. The results are long expressions and therefore two special cases are given instead. Both special cases assume identical layer properties. The first case assumes surface layers are dried such that $\Delta c_2 > 0$ but core layers are at eventual equilibrium moisture content or $\Delta c_1 = 0$. The transverse stresses in the layers become:

$$\overline{\sigma_{xx}^{(1r)}} = \frac{\big(\beta_{xx}Q_{xy} + \beta_{yy}(V_1Q_{yy} + V_2Q_{xx})\big)(Q_{xx}Q_{yy} - Q_{xy}^2)V_2}{V_1V_2(Q_{xx}^2 + Q_y^2) - Q_{xy}^2 + (V_1^2 + V_2^2)Q_{xx}Q_{yy}}\Delta c_2 \tag{73}$$

$$\overline{\sigma_{yy}^{(2r)}} = -\frac{\big(\beta_{yy}Q_{xy} + \beta_{xx}(V_1Q_{xx} + V_2Q_{yy})\big)(Q_{xx}Q_{yy} - Q_{xy}^2)V_1}{V_1V_2(Q_{xx}^2 + Q_y^2) - Q_{xy}^2 + (V_1^2 + V_2^2)Q_{xx}Q_{yy}}\Delta c_2 \tag{74}$$

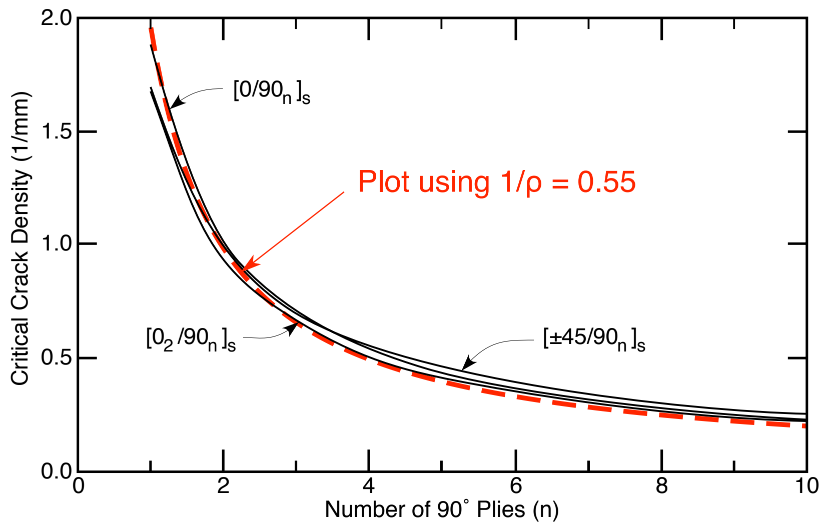

Fig. 2 The critical crack densities for delamination as functions of the number of 90°plies in half of the symmetric laminate. The three curves are for different (S) sublaminates supporting the 90° plies. The dashed red curve plots a prediction that onset of delamination reaches dimensionless crack spacing of 0.55 independent of layup type and layer thickness.

where superscripts on properties are dropped because both layers are the same. For the second case assume all layers are dried below equilibrium by equal amounts or $\Delta c_1 = \Delta c_2 > 0$. The transverse stresses in the layers become:

$$\overline{\sigma_{xx}^{(1r)}} = \frac{(\beta_{xx}-\beta_{yy})(V_1Q_{yy}+V_2Q_{xx}-Q_{xy})(Q_{xx}Q_{yy}-Q_{xy}^2)V_2}{V_1V_2(Q_{xx}^2+Q_y^2)-Q_{xy}^2+(V_1^2+V_2^2)Q_{xx}Q_{yy}}\Delta c_2 \tag{75}$$

$$\overline{\sigma_{yy}^{(2r)}} = -\frac{(\beta_{xx}-\beta_{yy})(V_1Q_{xx}+V_2Q_{yy}-Q_{xy})(Q_{xx}Q_{yy}-Q_{xy}^2)V_1}{V_1V_2(Q_{xx}^2+Q_y^2)-Q_{xy}^2+(V_1^2+V_2^2)Q_{xx}Q_{yy}}\Delta c_2 \tag{76}$$

For specific numbers, use the wood properties given in the paper and apply to a three-layer CLT panel ($V_1 = 1/3$ and $V_2 = 2/3$). If only surface layers are dried to $\Delta c_2 = 3\%$ below equilibrium core layers, the transverse stress in surface layers would be -4.21 MPa and in core layers would be +0.50 MPa. The surface layers do go into compression, but the core layers have tensile residual stresses that would promote cracking. If both layers are dried to $\Delta c_1 = \Delta c_2 = 3\%$ below panel equilibrium moisture constant, the transverse stress in surface layers would be -3.96 MPa and in core layers would be -4.18 MPa. Compared to drying only the surface layers, drying all layers gives about same compression in the surface and a beneficial level of compression in the core layers.

5 Onset of Delamination

The paper cites Nairn and Hu (1992) that if layer cracking toughness is equal to crack propagation toughness that delamination is predicted to start when dimensionless crack density is equal to $1/\rho = 0.55$. That paper actually never presents this number, but does plot the *dimensioned* crack density at the onset of delamination for several different layups as as a function of the thickness of the 90° layers. The dimensioned and dimensionless crack density are related by:

$$D = \frac{1}{2\rho t_1} \tag{77}$$

Thus, if we assume the critical dimensionless crack density is a constant (and equal to 0.55) and note that all calculations in Nairn and Hu (1992) used $2t_1 = 0.14$ mm for ply thickness, the layup independent prediction would be $D_{onset} = 0.55/(0.14 * 2 * n)$ where n is the number of plies in *half* the laminate (which is the x axis in Figure 5 from Nairn and Hu (1992)). Figure 2 superposes this new result on the calculations in Figure 5 from Nairn and Hu (1992). The results are very similar. In fact, it would have be better for Nairn and Hu (1992) to have interpreted results using dimensionless crack density, but that was not realized at the time.

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