Fracture Mechanics of Composites with Residual Thermal Stresses

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The problem of calculating the energy release rate for crack growth in an arbitrary composite in the presence of residual stresses is considered. First, a general expression is given for arbitrary, mixed traction and displacement boundary conditions. This general result is then applied to a series of specific problems including statistically homogeneous composites under traction or displacement boundary conditions, delamination of double cantilever beam specimens, and microcracking in the transverse plies of laminates. In many examples, the energy release rate in the presence of residual stresses can be reduced to finding the effect of damage on the effective mechanical properties of the composite. Because these effective properties can be evaluated by isothermal stress analysis, the effect of residual stresses on the energy release rate can be evaluated without recourse to any thermal elasticity stress analyses.

Introduction

When a composite comprised of materials having disparate thermal expansion coefficients is subjected to a uniform change in temperature, it will develop residual thermal stresses. If that composite develops cracks, either as a result of the residual thermal stresses or of some subsequently applied mechanical tractions, the residual and mechanical stresses will change to accommodate the fracture surfaces. One effect of the redistribution of residual thermal stresses is a change in the effective thermal expansion coefficient of the composite (Hashin, 1988). Additionally, the redistribution of residual and mechanical stresses releases strain energy that can drive crack propagation — the energy release rate for crack growth.

The goal of this paper is to evaluate the effect of residual thermal stresses on the energy release rate for crack growth in composites. The redistribution of residual and mechanical stresses caused by cracks can be viewed as similar elasticity problems but with different boundary conditions. Because of their similarity, it is often possible to explicitly account for the effect of residual thermal stresses without doing a thermal elasticity analysis of the composite. Some general energy release rate expressions including residual thermal stresses are derived in the first section of this paper. The general results are for an arbitrary composite under arbitrary, mixed traction and displacement boundary conditions. Next, the general expressions are applied to statistically homogeneous composites, double cantilever beam delamination specimens, and microcracked laminates. The simplest possible energy release rate expressions are derived. In some examples, the energy release depends on knowledge of both the mechanical and residual stress states, but, in some important practical examples, the energy release rate can be expressed in terms of the effect of cracks on the *effective* composite mechanical properties. These effective mechanical properties can be derived using elasticity analyses that ignore residual stresses.

General Theory

Consider an arbitrary, multiphase composite subjected a uniform temperature change of $\Delta T = T$ and to any mixed traction and displacement boundary conditions. Let the surface be $S = S_T \cup S_u$, where S_T is that part of the surface subjected to traction boundary conditions:

$$\vec{T} = \vec{T}^0 = \boldsymbol{\sigma}^0 \hat{n} \tag{1}$$

where $\boldsymbol{\sigma}^0$ is the surface applied stress (note: all tensors are written in bold-face) and $\hat{n}$ is a unit normal to the surface. S_u is that part of the surface subjected to displacement boundary conditions:

$$\vec{u} = \vec{u}^0 = \varepsilon^0 \vec{x} \tag{2}$$

where ε^0 is the surface strain and $\vec{x}$ is a position vector on the surface. $\vec{T}^0$ and $\vec{u}^0$ are the boundary conditions; in general they may depend on position. Note also that tractions and displacements may be applied over the same portions of the surface provided they are applied along different coordinates. For example, one section of the surface may have traction in the x direction but fixed displacement in the y direction. Similar to Rosen and Hashin (1970), the full thermal elasticity problem can be treated as a superposition of two problems. First, let $\boldsymbol{\sigma}^m$ and $\boldsymbol{\varepsilon}^m$, with tractions $\vec{T}^m$ and displacements $\vec{u}^m$, be the solution to the isothermal ($T = 0$) elasticity problem:

$$\boldsymbol{\sigma}^m \hat{n} = \vec{T}^0 \qquad \text{on} \qquad S_T \tag{3}$$

$$\vec{u}^m = \vec{u}^0 \qquad \text{on} \qquad S_u \tag{4}$$

$$\boldsymbol{\sigma}^m = \boldsymbol{C}\boldsymbol{\varepsilon}^m \qquad \text{and} \qquad \boldsymbol{\varepsilon}^m = \boldsymbol{S}\boldsymbol{\sigma}^m \tag{5}$$

where $\boldsymbol{C}$ and $\boldsymbol{S}$ are the stiffness and compliance tensors. In a composite, $\boldsymbol{C}$ and $\boldsymbol{S}$ will depend on position. As the solution to an elasticity problem, $\boldsymbol{\sigma}^m$ satisfies equilibrium and $\boldsymbol{\varepsilon}^m$ satisfies compatibility. Second, let $\boldsymbol{\sigma}^r$ and $\boldsymbol{\varepsilon}^r$, with tractions $\vec{T}^r$ and displacements $\vec{u}^r$, be the solution to the thermal elasticity problem:

$$\boldsymbol{\sigma}^r \hat{n} = 0 \qquad \text{on} \qquad S_T \tag{6}$$

$$\vec{u}^r = 0 \qquad \text{on} \qquad S_u \tag{7}$$

$$\boldsymbol{\sigma}^r = \boldsymbol{C}(\boldsymbol{\varepsilon}^r - \boldsymbol{\alpha}T) \qquad \text{and} \qquad \boldsymbol{\varepsilon}^r = \boldsymbol{S}\boldsymbol{\sigma}^r + \boldsymbol{\alpha}T \tag{8}$$

where $\boldsymbol{\alpha}$ is the position-dependent thermal expansion tensor, $\boldsymbol{\sigma}^r$ satisfies equilibrium, and $\boldsymbol{\varepsilon}^r$ satisfies compatibility. Note that $\vec{T}^r$ may be nonzero on S_u and $\vec{u}^r$ may be nonzero on S_T. A superposition of the above two solutions, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^m + \boldsymbol{\sigma}^r$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^m + \boldsymbol{\varepsilon}^r$, solves the complete thermal elasticity problem

$$\boldsymbol{\sigma}\hat{n} = \vec{T}^0 \qquad \text{on} \qquad S_T \tag{9}$$

$$\vec{u} = \vec{u}^0 \qquad \text{on} \qquad S_u \tag{10}$$

$$\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}T) \qquad \text{and} \qquad \boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma} + \boldsymbol{\alpha}T \tag{11}$$

$\boldsymbol{\sigma}^m$ and $\boldsymbol{\varepsilon}^m$ are the mechanical stresses and strains while $\boldsymbol{\sigma}^r$ and $\boldsymbol{\varepsilon}^r$ are the residual thermal stresses and strains.

The internal energy is the volume integral of stress dotted into strain relative to the free-expansion strains. This energy can be expressed in terms of mechanical and residual stresses:

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\alpha}T)dV = \frac{1}{2}\int_V \boldsymbol{\sigma}^m \boldsymbol{S}\boldsymbol{\sigma}^m dV + \int_V \boldsymbol{\sigma}^m \boldsymbol{S}\boldsymbol{\sigma}^r dV + \frac{1}{2}\int_V \boldsymbol{\sigma}^r \boldsymbol{S}\boldsymbol{\sigma}^r dV \tag{12}$$

Assuming traction-free crack surfaces and using the divergence theorem, the mechanical stress term in Eq. (12) may be written in terms of mechanical surface tractions, $\vec{T}^m$, and displacements, $\vec{u}^m$, as

$$\frac{1}{2}\int_V \boldsymbol{\sigma}^m \boldsymbol{S}\boldsymbol{\sigma}^m dV = \frac{1}{2}\int_{S_T} \vec{T}^0 \cdot \vec{u}^m dS + \frac{1}{2}\int_{S_u} \vec{T}^m \cdot \vec{u}^0 dS \tag{13}$$

For the cross term in Eq. (12), the displacements caused by the residual thermal stresses can be considered as virtual displacements to the mechanical strains or $\delta\vec{u} = \vec{u}^r$. They correctly have $\vec{u}^r = 0$ on S_u. By virtual work

$$\int_V \boldsymbol{\sigma}^m \cdot \boldsymbol{\varepsilon}^r dV = \int_{S_T} \vec{T}^0 \cdot \vec{u}^r dS \tag{14}$$

But also, $\boldsymbol{\varepsilon}^r = \boldsymbol{S}\boldsymbol{\sigma}^r + \boldsymbol{\alpha}T$, giving

$$\int_V \boldsymbol{\sigma}^m \boldsymbol{S}\boldsymbol{\sigma}^r dV = \int_{S_T} \vec{T}^0 \cdot \vec{u}^r dS - \int_V \boldsymbol{\sigma}^m \cdot \boldsymbol{\alpha}T dV \tag{15}$$

Analogously, the residual tractions, $\vec{T}^r$, can be considered as virtual forces on the mechanical stresses; $\boldsymbol{\sigma}^r$ obeys equilibrium and $\vec{T}^r = 0$ on S_T. Thus, again by virtual work

$$\int_V \boldsymbol{\varepsilon}^m \cdot \boldsymbol{\sigma}^r dV = \int_V \boldsymbol{\sigma}^m \boldsymbol{S}\boldsymbol{\sigma}^r dV = \int_{S_u} \vec{T}^r \cdot \vec{u}^0 dS \tag{16}$$

Finally, for the pure residual stress term in Eq. (12):

$$\frac{1}{2}\int_V \boldsymbol{\sigma}^r \boldsymbol{S} \boldsymbol{\sigma}^r dV = \frac{1}{2}\int_V (\boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r - \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha}T)dV = \frac{1}{2}\left(\int_S \vec{T}^r \cdot \vec{u}^r dS - \int_V \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha}T dV\right) \qquad (17)$$

But, over the entire surface, either $\vec{T}^r = 0$ or $\vec{u}^r = 0$, giving

$$\frac{1}{2}\int_V \boldsymbol{\sigma}^r \boldsymbol{S} \boldsymbol{\sigma}^r dV = -\frac{1}{2}\int_V \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha}T dV \qquad (18)$$

Combining Eqs. (13), (15), (16), and (18) gives

$$U = \frac{1}{2}\int_{S_T} \vec{T}^0 \cdot \vec{u}^m dS + \frac{1}{2}\int_{S_u} \vec{T}^m \cdot \vec{u}^0 dS + \int_{S_u} \vec{T}^r \cdot \vec{u}^0 dS - \frac{1}{2}\int_V \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha}T dV \qquad (19)$$

$$\int_V \boldsymbol{\sigma}^m \cdot \boldsymbol{\alpha}T = \int_{S_T} \vec{T}^0 \cdot \vec{u}^r dS - \int_{S_u} \vec{T}^r \cdot \vec{u}^0 dS \qquad (20)$$

For fracture mechanics analysis we need to evaluate the energy release rate defined as

$$G = \frac{dW}{dA} - \frac{dU}{dA} \qquad (21)$$

where W is external work and A is fracture area. The external work is caused by the action of the mechanical tractions on S_T applied over the *total* displacements. Thus

$$\frac{dW}{dA} = \frac{d}{dA}\left(\int_{S_T} \vec{T}^0 \cdot (\vec{u}^m + \vec{u}^r)dS\right) \qquad (22)$$

Combining this result with Eqs. (19) and (20) gives

$$G = G_{mech} + \frac{VT}{2}\left(2\frac{d\langle\boldsymbol{\sigma}^m \cdot \boldsymbol{\alpha}\rangle}{dA} + \frac{d\langle\boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha}\rangle}{dA}\right) \qquad (23)$$

where G_{mech} is the mechanical energy release rate or the energy release when $T = 0$:

$$G_{mech} = \frac{d}{dA}\left(\frac{1}{2}\int_{S_T} \vec{T}^0 \cdot \vec{u}^m dS - \frac{1}{2}\int_{S_u} \vec{T}^m \cdot \vec{u}^0 dS\right) \qquad (24)$$

and angle brackets indicates a volume-averaged quantity:

$$\langle f(x,y,z)\rangle = \frac{1}{V}\int_V f(x,y,z)dV \qquad (25)$$

Note that the full, thermodynamic internal energy in thermal elasticity includes heat capacity terms. The energy associated with residual stresses that exist under zero traction (or zero displacement), can be viewed as contributing to the effective constant pressure (or constant volume) heat capacity of the composite (Rosen and Hashin, 1970). Here the residual stress energy contributes to the total energy release rate for crack growth. The heat capacity terms associated with individual phases, however, can be ignored when calculating energy release rates because they drop out when the derivative with respect to crack area is evaluated.

The result in Eq. (23) includes nothing specific about composites. The result for a composite with n phases is

$$G = G_{mech} + \frac{VT}{2}\sum_{i=1}^n v_i \boldsymbol{\alpha}^{(i)} \cdot \left(2\frac{d\overline{\boldsymbol{\sigma}^{(im)}}}{dA} + \frac{d\overline{\boldsymbol{\sigma}^{(ir)}}}{dA}\right) \qquad (26)$$

where superscript (i) means a property of phase i, v_i is the volume fraction of phase i, and the over bar indicates a phase-averaged quantity integrated over the volume of phase i, V_i:

$$\overline{\boldsymbol{\sigma}^{(im)}} = \frac{1}{Vv_i}\int_{V_i} \boldsymbol{\sigma}^m dV \qquad (27)$$

In deriving Eq. (26), it was assumed that the component thermal expansion tensors in each phase are independent of position within that phase. This situation always holds for composites with isotropic phases. For composites with anisotropic phases at variable orientations, each orientation must be considered as a separate phase for analysis by Eq. (26). When anisotropic phases are aligned, such as in unidirectional, carbon-fiber reinforced composites, the anisotropic phase can be considered as a single phase.

Although the general result in Eq. (26) depends on both mechanical and thermal stresses, in some important special cases, the energy release rate can be expressed solely in terms of mechanical properties and stresses. For example, in many fracture situations, cracks will propagate such that failure is by pure mode I fracture or pure opening mode fracture. For example, it has been suggested that regardless of external loading conditions that delamination always proceeds by mode I fracture (Charalambides et al., 1992). Under pure mode I crack growth, energy release rate is related to mode I stress intensity factor by

$$G_I \propto K_I^2 \tag{28}$$

In linear thermal elasticity, K_I must be linearly related to applied load and temperature change. If we thus consider uniformly scaling all applied tractions and displacements to be

$$\vec{T}^{app} = P\vec{T}^0 \qquad \text{and} \qquad \vec{u}^{app} = P\vec{u}^0 \tag{29}$$

where P is a scalar multiplier, stress intensity must have the form

$$K_I = c_1 P + c_2 T \tag{30}$$

where c_1 and c_2 are some constants that depend on material properties and geometry of the crack. The energy release rate must therefore have the form

$$G_I \propto (c_1 P + c_2 T)^2 \tag{31}$$

Comparing Eq. (31) to Eq. (26) we can prove that

$$\frac{TV}{2} \frac{\sum_{i=1}^n v_i \boldsymbol{\alpha}^{(i)} \cdot \frac{\overline{d\boldsymbol{\sigma}^{(im)}}}{dA}}{G_{mech}} = \frac{\sum_{i=1}^n v_i \boldsymbol{\alpha}^{(i)} \cdot \frac{\overline{d\boldsymbol{\sigma}^{(ir)}}}{dA}}{\sum_{i=1}^n v_i \boldsymbol{\alpha}^{(i)} \cdot \frac{\overline{d\boldsymbol{\sigma}^{(im)}}}{dA}} \tag{32}$$

Substitution back into Eq. (26) gives

$$\frac{G_I}{G_{mech}} = \left(1 + \frac{TV}{2} \frac{\sum_{i=1}^n v_i \boldsymbol{\alpha}^{(i)} \cdot \frac{\overline{d\boldsymbol{\sigma}^{(im)}}}{dA}}{G_{mech}} \right)^2 \tag{33}$$

Remarkably, the energy release rate including residual thermal stresses depends only on the results from a mechanical stress analysis. Thus, the effect of residual stresses can be determined without any need for solving a thermal elasticity problem. Although this result was derived for pure mode I crack growth, analogous expressions hold for pure mode II or III crack growth.

Examples

Uniform Traction, Statistically Isotropic Composite

A useful model for evaluating effective properties of composite materials is to consider a statistically homogeneous composite (Hashin, 1963; 1969). Such composites are analyzed by averaging over a representative volume element; this element must be chosen large enough to have all averaged results representative of the averaged results for entire composite (Hashin, 1963; 1969). Here we also consider the representative volume element to have cracks that are representative of the entire composite. Such a representative volume element is then subjected to a *uniform* traction or a traction that arises from a surface stress, $\boldsymbol{\sigma}^0$, that is independent

of position. By the average strain theorem (Hill, 1963; Hashin 1963; 1969), the only remaining integral in G_{mech} reduces to

$$\frac{1}{2}\int_S \vec{T}^0 \cdot \vec{u}^m dS = \frac{V}{2}\boldsymbol{\sigma}^0 \cdot (\langle \boldsymbol{\varepsilon}^m \rangle - \boldsymbol{D}^m) = \frac{V}{2}\boldsymbol{\sigma}^0 \cdot \boldsymbol{S}^* \boldsymbol{\sigma}^0 \tag{34}$$

where $\boldsymbol{S}^*$ is the effective compliance tensor of the composite (Hashin, 1963; 1969). The tensor $\boldsymbol{D}^m$ is related to the crack opening mechanical displacements and is defined by

$$D^m_{ij} = \frac{1}{2V}\int_{S_c} \left(u^m_j n_i + u^m_i n_j\right) dS \qquad (i, j = x,\ y,\ \text{or } z) \tag{35}$$

where S_c is total crack surface area. The effective composite compliance is defined in terms of the *apparent* average strain which is the bulk average strain reduced by the crack-opening tensor (Aboudi, 1994); thus

$$\boldsymbol{S}^* \boldsymbol{\sigma}^0 = \langle \boldsymbol{\varepsilon}^m \rangle - \boldsymbol{D}^m \tag{36}$$

Substituting Eq. (34) into Eq. (26) the energy release rate becomes

$$G = \frac{V}{2}\left[\boldsymbol{\sigma}^0 \cdot \frac{d\boldsymbol{S}^*}{dA}\boldsymbol{\sigma}^0 + T\sum_{i=1}^n v_i \boldsymbol{\alpha}^{(i)} \cdot \left(2\frac{d\overline{\boldsymbol{\sigma}^{(im)}}}{dA} + \frac{d\overline{\boldsymbol{\sigma}^{(ir)}}}{dA}\right)\right] \tag{37}$$

For pure mode I (or II, or III) crack growth the energy release rate becomes

$$\frac{G_I}{G_{mech}} = \left(1 + T\frac{\sum_{i=1}^n v_i \boldsymbol{\alpha}^{(i)} \cdot \dfrac{d\overline{\boldsymbol{\sigma}^{(im)}}}{dA}}{\boldsymbol{\sigma}^0 \cdot \dfrac{d\boldsymbol{S}^*}{dA}\boldsymbol{\sigma}^0}\right)^2 \tag{38}$$

Finally, for a two phase composite (a composite with two isotropic phases or two aligned anisotropic phases) we have

$$v_1 \frac{d\overline{\boldsymbol{\sigma}^{(1m)}}}{dA} + v_2 \frac{d\overline{\boldsymbol{\sigma}^{(2m)}}}{dA} = 0 \tag{39}$$

$$\boldsymbol{S}^{(1)}v_1 \frac{d\overline{\boldsymbol{\sigma}^{(1m)}}}{dA} + \boldsymbol{S}^{(2)}v_2 \frac{d\overline{\boldsymbol{\sigma}^{(2m)}}}{dA} = \frac{d\langle \boldsymbol{\varepsilon}^m \rangle}{dA} = \frac{d\boldsymbol{S}^*}{dA}\boldsymbol{\sigma}^0 + \frac{d\boldsymbol{D}^m}{dA} \tag{40}$$

For pure mode fracture we can derive

$$G_I = \frac{V}{2}\frac{\left[\left(\boldsymbol{\sigma}^0 + T(\boldsymbol{\alpha}^{(1)} - \boldsymbol{\alpha}^{(2)})(\boldsymbol{S}^{(1)} - \boldsymbol{S}^{(2)})^{-1}\right) \cdot \dfrac{d\boldsymbol{S}^*}{dA}\boldsymbol{\sigma}^0 + T(\boldsymbol{\alpha}^{(1)} - \boldsymbol{\alpha}^{(2)})(\boldsymbol{S}^{(1)} - \boldsymbol{S}^{(2)})^{-1} \cdot \dfrac{d\boldsymbol{D}^m}{dA}\right]^2}{\boldsymbol{\sigma}^0 \cdot \dfrac{d\boldsymbol{S}^*}{dA}\boldsymbol{\sigma}^0} \tag{41}$$

The energy release rate depends only on the effective compliance tensor and the mechanically-induced crack-opening displacement tensor. Note the $\boldsymbol{D}^m$ only affects the energy release rate when there are residual thermal stresses; in the absence of residual stresses, G_I depends only on $\boldsymbol{S}^*$.

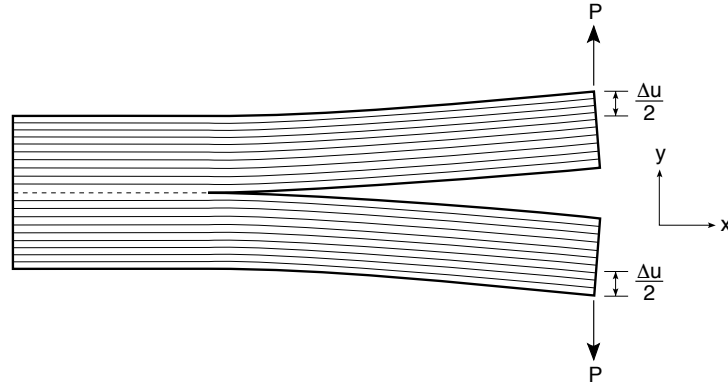*Uniform Displacement, Statistically Isotropic Composite*

For uniform displacement ($\vec{u}^0 = \boldsymbol{\varepsilon}^0 \vec{x}$ where $\boldsymbol{\varepsilon}^0$ is independent of position) applied to a statistically homogeneous composite, use of the average stress theorem leads to

$$G_{mech} = -\frac{V}{2}\frac{d\langle \boldsymbol{\sigma}^m \rangle}{dA} \cdot \boldsymbol{\varepsilon}^0 = -\frac{V}{2}\boldsymbol{\varepsilon}^0 \cdot \frac{d\boldsymbol{C}^*}{dA} \cdot \boldsymbol{\varepsilon}^0 \tag{42}$$

where $\boldsymbol{C}^*$ is the effective stiffness tensor of the composite. For a better analogy with the uniform traction results, some terms can be re-expressed as:

$$\boldsymbol{\alpha}^{(i)} \cdot \overline{\boldsymbol{\sigma}^{(im)}} = -\boldsymbol{M}^{(i)} \cdot \overline{\boldsymbol{\varepsilon}^{(im)}} \tag{43}$$

$$\boldsymbol{\alpha}^{(i)} \cdot \overline{\boldsymbol{\sigma}^{(ir)}} = -\boldsymbol{M}^{(i)} \cdot \overline{\boldsymbol{\varepsilon}^{(ir)}} + \boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{M}^{(i)}T \tag{44}$$

Fig. 1 A double cantilever beam specimen using unidirectional composites. The fiber direction (indicated by the internal lines) is parallel to the delamination crack growth. The specimen is point loaded by loads of P. The total deflection between the loading points is Δu.

where $\boldsymbol{M} = -\boldsymbol{C}\boldsymbol{\alpha}$ is the stress-temperature tensor. Incorporating these results into the Eq. (26) and ignoring the constant term $\boldsymbol{\alpha}^{(i)} \cdot \boldsymbol{M}^{(i)} T$, which drops out upon differentiation, the energy release rate is

$$G = -\frac{V}{2}\left[\varepsilon^0 \cdot \frac{d\boldsymbol{C}^*}{dA}\varepsilon^0 + T\sum_{i=1}^{n} v_i \boldsymbol{M}^{(i)} \cdot \left(2\frac{d\overline{\varepsilon^{(im)}}}{dA} + \frac{d\overline{\varepsilon^{(ir)}}}{dA}\right)\right] \tag{45}$$

This result is identical to the uniform traction results (see Eq. (37)) except for a sign change and physical meanings of all symbols. The sign change is required because G is positive and, in general, $\boldsymbol{C}^*$ decreases with crack growth. By an explicit analogy to the results for uniform traction, the energy release rate for pure mode crack growth becomes

$$\frac{G_I}{G_{mech}} = \left(1 + T\frac{\sum_{i=1}^{n} v_i \boldsymbol{M}^{(i)} \cdot \frac{d\overline{\varepsilon^{(im)}}}{dA}}{\varepsilon^0 \cdot \frac{d\boldsymbol{C}^*}{dA}\varepsilon^0}\right)^2 \tag{46}$$

Delamination

Figure 1 shows a double cantilever beam specimen made from a unidirectional composite. The fibers run parallel to the x direction. This problem differs from the previous sections in two respects. First, the specimen is not statistically homogeneous because there is only one macroscopic delamination crack. Second, although there is only traction loading, the tractions are not uniform (*i.e.*, they depend on position). The tractions can be written as

$$\vec{T}^0 = P\big[\delta(\vec{x} - \vec{x}_{upper}) - \delta(\vec{x} - \vec{x}_{lower})\big] \tag{47}$$
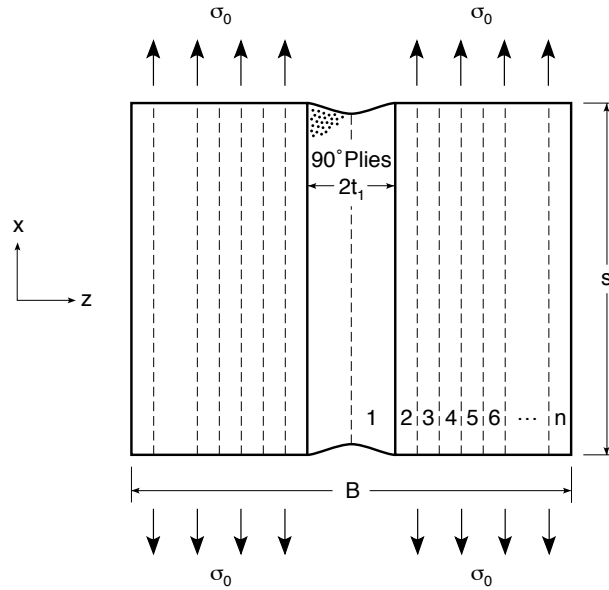
where $\delta(\vec{x})$ is the Dirac delta function and $\vec{x}_{upper}$ and $\vec{x}_{lower}$ are the locations of the upper and lower load points. Substitution into Eqs. (24) and (33) (because the crack growth is pure mode I) gives

$$G_I = \frac{P^2}{2}\frac{dC_m}{dA}\left(1 + \frac{TV}{2}\frac{\sum_{i=1}^{n} v_i\boldsymbol{\alpha}^{(i)} \cdot \frac{d\overline{\boldsymbol{\sigma}^{(im)}}}{dA}}{\frac{P^2}{2}\frac{dC_m}{dA}}\right)^2 \tag{48}$$

where $C_m = \Delta u^m/P$ is the mechanical load-point compliance. The leading term is G_{mech} or the energy release rate in the absence of residual stresses

   Proceeding for a two-phase unidirectional composite (fibers as phase 1 and matrix as phase 2) with aligned transversely isotropic phases, we can write

$$\sum_{i=1}^{n} v_i\boldsymbol{\alpha}^{(i)} \cdot \frac{d\overline{\boldsymbol{\sigma}^{(im)}}}{dA} = v_1\left[\overline{\sigma_{xx}^{(1m)}}\alpha_A^{(1)} + \left(\overline{\sigma_{yy}^{(1m)}} + \overline{\sigma_{zz}^{(1m)}}\right)\alpha_T^{(1)}\right] + v_2\left[\overline{\sigma_{xx}^{(2m)}}\alpha_A^{(2)} + \left(\overline{\sigma_{yy}^{(2m)}} + \overline{\sigma_{zz}^{(2m)}}\right)\alpha_T^{(2)}\right] \tag{49}$$

Fig. 2 An edge view of a microcracked laminate. The microcracks occur in the central 90° plies; they are located at the top and bottom of the 90° plies are are separated by a distance s. Each ply has thickness t_i; the total symmetric laminate thickness is B.

Now, double-cantilever beams specimens without residual stresses can be well analyzed using beam analyses (Hashemi et al., 1990). Within the assumptions of a beam analysis $\overline{\sigma_{yy}^{(im)}}$ and $\overline{\sigma_{zz}^{(im)}}$ are identically zero while $\overline{\sigma_{xx}^{(im)}}$ will net to zero by virtue of a balance between tension and compression stresses within the beam. The energy release rate for delamination becomes simply

$$G = \frac{P^2}{2} \frac{dC_m}{dA} \tag{50}$$

In other words, within the assumptions of beam analyses, residual stresses have no effect on delamination. Residual stresses are present in both the fiber and the matrix, but they release no energy as the delamination propagates.

Microcracked Laminate

Figure 2 shows an edge view of a symmetric, $2n$-ply laminate under a uniaxial load of σ_0 in the x direction in which the central 90° ply group has through-the-width cracks separated by a distance s. This geometry is appropriate for analysis of a microcracked laminate with periodic microcracks in the 90° plies having a density $D = 1/s$ (Nairn and Hu, 1994). This problem differs from previous examples in three respects. First, the tractions are nonuniform: σ_0 is applied to the uncracked plies, but there is zero traction on the cracked 90° plies. Second, there are more then two phases. Although the plies themselves are composites, here each ply is treated as a homogeneous, anisotropic material. The general laminate, which may have plies at many orientations or may even be a hybrid laminate, is thus a multiphase composite. Third, we incorporate information about the laminated structure of the composite into the analysis. The key structural information we use is that the x and y direction strains of all uncracked plies are equal and that $\overline{\sigma_{zz}^{(im)}} = \overline{\sigma_{zz}^{(ir)}} = 0$ in all plies.

From Eq. (23), we can immediately write

$$G = \frac{V}{2} \frac{d}{dA} \left(\frac{\sigma_0^2}{E_{xx}^*} + T \big(2 \langle \boldsymbol{\sigma}^m \cdot \boldsymbol{\alpha} \rangle + \langle \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha} \rangle \big) \right) \tag{51}$$

where E_{xx}^* is the effective laminate modulus in the x direction. The first term is the energy release rate in the absence of residual stresses. Furthermore, by Eq. (20) we can write

$$\langle \boldsymbol{\sigma}^m \cdot \boldsymbol{\alpha} T \rangle = \sigma_0 \langle \varepsilon_{xx}^r \rangle_u = \sigma_0 \alpha_{xx}^* T \tag{52}$$

where α_{xx}^* is the effective laminate thermal expansion coefficient in the x direction and $\langle \varepsilon_{xx}^r \rangle_u$ is the volume averaged residual strain in the *uncracked* plies only. This term appears because the tractions are applied only to the uncracked plies. Equation (52) is similar to Levin's analysis for thermal expansion coefficient of a composite (Levin, 1967; Rosen and Hashin, 1970) except that it is applied here under conditions of nonuniform traction.

The energy release rate will be analyzed first using a plane stress analysis in which $\overline{\sigma_{yy}^{(im)}} = \overline{\sigma_{yy}^{(ir)}} = 0$ in all plies. From Eq. (52):

$$\sigma_0 \alpha_{xx}^* = \sum_{i=1}^{n} v_i \overline{\sigma_{xx}^{(im)}} \alpha_{xx}^{(i)} \tag{53}$$

where $v_i = 2t_i/B$ is the total volume fraction of ply i including both sides of the symmetric laminate. By force balance and iso-strain in the uncracked plies

$$\sum_{i=1}^{n} v_i \overline{\sigma_{xx}^{(im)}} = \sigma_0 \qquad \text{and} \qquad \overline{\sigma_{xx}^{(im)}} = \frac{E_{xx}^{(i)} \sigma_0}{E_{xx}^*} \quad \text{for } i \neq 1 \tag{54}$$

Using these results, the effective x-direction thermal expansion coefficient can be expressed as

$$\alpha_{xx}^* = \alpha_{xx}^{(i)} - \frac{(1 - v_1) \Delta \alpha_{xx} E_{xx}^{(u)}}{E_{xx}^*} \tag{55}$$

where

$$\Delta \alpha_{xx} = \alpha_{xx}^{(1)} - \alpha_{xx}^{(u)}, \qquad E_{xx}^{(u)} = \frac{\sum_{i=2}^{n} v_i E_{xx}^{(i)}}{1 - v_1}, \qquad \text{and} \qquad \alpha_{xx}^{(u)} = \frac{\sum_{i=2}^{n} v_i \alpha_{xx}^{(i)} E_{xx}^{(i)}}{\sum_{i=2}^{n} v_i E_{xx}^{(i)}} \tag{56}$$

$E_{xx}^{(u)}$ and $\alpha_{xx}^{(u)}$ are the rule-of-mixtures x-direction modulus and thermal expansion coefficient of the uncracked plies. The residual strain energy term is

$$\langle \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha} T \rangle = \sum_{i=1}^{n} v_i \overline{\sigma_{xx}^{(ir)}} \alpha_{xx}^{(i)} T \tag{57}$$

By force balance of residual stresses and iso-strain in the uncracked plies

$$\sum_{i=1}^{n} v_i \overline{\sigma_{xx}^{(ir)}} = o \qquad \text{and} \qquad \overline{\sigma_{xx}^{(ir)}} = E_{xx}^{(i)} \left(\alpha_{xx}^* - \alpha_{xx}^{(i)} \right) T \quad \text{for } i \neq 1 \tag{58}$$

Using these results, the residual strain energy term is

$$\langle \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha} T \rangle = \text{constant} - \alpha_{xx}^* (1 - v_1) T^2 \Delta \alpha_{xx} E_{xx}^{(u)} \tag{59}$$

The "constant" term is not evaluated because it drops out when finding the energy release rate.

Substituting Eqs. (52), (55), and (59) into Eq. (51) gives

$$G = \frac{V}{2} \left(\sigma_0 - \Delta \alpha_{xx} E_{xx}^{(u)} (1 - v_1) T \right)^2 \frac{d}{dA} \left(\frac{1}{E_{xx}^*} \right) \tag{60}$$

The energy release rate in the presence of residual thermal stresses depends only on one effective property — E_{xx}^*; that property can be evaluated by an isothermal stress analysis. If we imagine formation of a complete new microcrack midway between two existing microcracks, we have

$$V = BWs = \frac{BW}{D} \qquad \text{and} \qquad dA = 2t_1 W = v_1 BW \tag{61}$$

The integrated energy release rate for formation of a complete microcrack is

$$G = \frac{1}{2v_1 D} \left(\sigma_0 - \Delta\alpha_{xx} E_{xx}^{(u)} (1 - v_1) T \right)^2 \Delta \left(\frac{1}{E_{xx}^*} \right) \tag{62}$$

where $\Delta(1/E_{xx}^*)$ is the change in the effective x-direction modulus caused by the new microcrack. An energy release expression mathematically identical to Eq. (62) is given in a different form in Nairn et al. (1993) and Nairn and Hu (1994) and in the identical form in Nairn (1995). In Nairn et al. (1993), Nairn and Hu (1994), and Nairn (1995), however, the energy release rate was the specific result of a thermal elasticity analysis. Here the energy release rate is derived as a general result. The accuracy of finding G is only limited to the accuracy in finding E_{xx}^*. In Nairn (1995), variational mechanics was used to bound E_{xx}^* and an average of those bounds gave a virtually exact solution for G.

A better connection with laminated plate theory can be derived by considering a generalized plane-strain analysis of a cracked laminate instead of the previous plane stress analysis. In the generalized plane-strain analysis, it is assumed that the y-direction strain in all plies is $-\nu_{xy}^* \sigma_0 / E_{xx}^*$ where ν_{xy}^* is the effective x-y Poisson's ratio of the cracked laminate. The resulting new analysis will correct the plane stress analysis to account for differential Poisson's ratios between the plies. The corrections are generally small, but can be accomplished in a straight-forward analysis.

Equations (51) and (52) are unchanged in the generalized plane-strain analysis. Starting from Eq. (52):

$$\sigma_0 \alpha_{xx}^* = \sum_{i=1}^{n} \left(v_i \overline{\sigma_{xx}^{(im)}} \alpha_{xx}^{(i)} + v_i \overline{\sigma_{yy}^{(im)}} \alpha_{yy}^{(i)} \right) \tag{63}$$

By force balance and iso-strain in the uncracked plies

$$\sum_{i=1}^{n} v_i \overline{\sigma_{xx}^{(im)}} = \sigma_0 \qquad \text{and} \qquad \sum_{i=1}^{n} v_i \overline{\sigma_{yy}^{(im)}} = 0 \tag{64}$$

$$\overline{\sigma_{xx}^{(im)}} = \left(Q_{11}^{(i)} - \nu_{xy}^* Q_{12}^{(i)} \right) \frac{\sigma_0}{E_{xx}^*} \qquad \text{for } i \neq 1 \tag{65}$$

$$\overline{\sigma_{yy}^{(im)}} = \left(Q_{12}^{(i)} - \nu_{xy}^* Q_{22}^{(i)} \right) \frac{\sigma_0}{E_{xx}^*} \qquad \text{for } i \neq 1 \tag{66}$$

where $Q_{kj}^{(i)}$ $(k, j = 1, 2, 6)$ are the suitably-rotated elements of the in-plane stiffness matrix for ply i (Tsai, 1987). Using these results, the effective x-direction thermal expansion coefficient becomes

$$\begin{aligned}
\alpha_{xx}^* = {} & \alpha_{xx}^{(1)} + \frac{1}{E_{xx}^*} \sum_{i=2}^{n} v_i \left[\left(\alpha_{xx}^{(i)} - \alpha_{xx}^{(1)} \right) Q_{11}^{(i)} + \left(\alpha_{yy}^{(i)} - \alpha_{yy}^{(1)} \right) Q_{12}^{(i)} \right] \\
& - \frac{\nu_{xy}^*}{E_{xx}^*} \sum_{i=2}^{n} v_i \left[\left(\alpha_{xx}^{(i)} - \alpha_{xx}^{(1)} \right) Q_{12}^{(i)} + \left(\alpha_{yy}^{(i)} - \alpha_{yy}^{(1)} \right) Q_{22}^{(i)} \right]
\end{aligned} \tag{67}$$

Now consider the laminate created by removing the cracked plies and joining the remaining plies; call it the "uncracked" laminate. By laminated plate analysis of the *symmetric* "uncracked" laminate (Tsai, 1987), it can be shown that

$$\sum_{i=2}^{n} v_i Q_{kj}^{(i)} = \frac{A_{kj}^{(u)}}{B} \tag{68}$$

$$\sum_{i=2}^{n} v_i \left(\alpha_{xx}^{(i)} Q_{11}^{(i)} + \alpha_{yy}^{(i)} Q_{12}^{(i)} \right) = \frac{1}{B} \left(A_{11}^{(u)} \alpha_{xx}^{(u)} + A_{12}^{(u)} \alpha_{yy}^{(u)} \right) \tag{69}$$

$$\sum_{i=2}^{n} v_i \left(\alpha_{xx}^{(i)} Q_{12}^{(i)} + \alpha_{yy}^{(i)} Q_{22}^{(i)} \right) = \frac{1}{B} \left(A_{12}^{(u)} \alpha_{xx}^{(u)} + A_{22}^{(u)} \alpha_{yy}^{(u)} \right) \tag{70}$$

where $A_{kj}^{(u)}$ are the elements of the laminated plate theory $[A]$ matrix for the "uncracked" laminate and $\alpha_{xx}^{(u)}$ and $\alpha_{yy}^{(u)}$ are the thermal expansion coefficients for the "uncracked" laminate defined by:

$$\alpha_{xx}^{(u)} = B \left[a_{11}^{(u)} \sum_{i=2}^{n} v_i \left(\alpha_{xx}^{(i)} Q_{11}^{(i)} + \alpha_{yy}^{(i)} Q_{12}^{(i)} \right) + a_{12}^{(u)} \sum_{i=2}^{n} v_i \left(\alpha_{xx}^{(i)} Q_{12}^{(i)} + \alpha_{yy}^{(i)} Q_{22}^{(i)} \right) \right] \tag{71}$$

$$\alpha_{yy}^{(u)} = B \left[a_{12}^{(u)} \sum_{i=2}^{n} v_i \left(\alpha_{xx}^{(i)} Q_{11}^{(i)} + \alpha_{yy}^{(i)} Q_{12}^{(i)} \right) + a_{22}^{(u)} \sum_{i=2}^{n} v_i \left(\alpha_{xx}^{(i)} Q_{12}^{(i)} + \alpha_{yy}^{(i)} Q_{22}^{(i)} \right) \right] \tag{72}$$

Here $a_{kj}^{(u)}$ are elements of the laminated plate theory $[a]$ matrix which is the inverse of the $[A]$ matrix. Using these results, the effective x-direction thermal expansion coefficient can be expressed as

$$\alpha_{xx}^* = \alpha_{xx}^{(1)} - \frac{(1 - v_1)\Delta_1}{E_{xx}^*} + \frac{\nu_{xy}^*(1 - v_1)\Delta_2}{E_{xx}^*} \tag{73}$$

where

$$\Delta_1 = \frac{1}{B(1 - v_1)} \left[A_{11}^{(u)} \left(\alpha_{xx}^{(1)} - \alpha_{xx}^{(u)} \right) + A_{12}^{(u)} \left(\alpha_{yy}^{(1)} - \alpha_{yy}^{(u)} \right) \right] \tag{74}$$

$$\Delta_2 = \frac{1}{B(1 - v_1)} \left[A_{12}^{(u)} \left(\alpha_{xx}^{(1)} - \alpha_{xx}^{(u)} \right) + A_{22}^{(u)} \left(\alpha_{yy}^{(1)} - \alpha_{yy}^{(u)} \right) \right] \tag{75}$$

Note that $A_{kj}^{(u)}/(B(1-v_1))$ are the elements of the normalized, $x-y$ plane stiffness matrix for the "uncracked" laminate (Tsai, 1987). By a similar analysis but with σ_0 applied in the y direction, the effective y-direction thermal expansion coefficient can be expressed as

$$\alpha_{yy}^* = \alpha_{yy}^{(1)} - \frac{(1 - v_1)\Delta_2}{E_{yy}^*} + \frac{\nu_{xy}^*(1 - v_1)\Delta_1}{E_{xx}^*} \tag{76}$$

The residual strain energy term is

$$\langle \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha}T \rangle = \sum_{i=1}^{n} \left(v_i \overline{\sigma_{xx}^{(ir)}} \alpha_{xx}^{(i)} T + v_i \overline{\sigma_{yy}^{(ir)}} \alpha_{yy}^{(i)} T \right) \tag{77}$$

By force balance of residual stresses and iso-strain in the uncracked plies

$$\sum_{i=1}^{n} v_i \overline{\sigma_{xx}^{(ir)}} = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} v_i \overline{\sigma_{yy}^{(ir)}} = 0 \tag{78}$$

$$\overline{\sigma_{xx}^{(ir)}} = Q_{11}^{(i)} \left(\alpha_{xx}^* - \alpha_{xx}^{(i)} \right) T + Q_{12}^{(i)} \left(\alpha_{yy}^* - \alpha_{yy}^{(i)} \right) T \qquad \text{for } i \neq 1 \tag{79}$$

$$\overline{\sigma_{yy}^{(ir)}} = Q_{12}^{(i)} \left(\alpha_{xx}^* - \alpha_{xx}^{(i)} \right) T + Q_{22}^{(i)} \left(\alpha_{yy}^* - \alpha_{yy}^{(i)} \right) T \qquad \text{for } i \neq 1 \tag{80}$$

Using these results, the residual strain energy term is

$$\langle \boldsymbol{\sigma}^r \cdot \boldsymbol{\alpha}T \rangle = \text{constant} - \alpha_{xx}^*(1 - v_1)T^2\Delta_1 - \alpha_{yy}^*(1 - v_1)T^2\Delta_2 \tag{81}$$

The "constant" term is not evaluated because it drops out when finding the energy release rate.

Substituting Eqs. (52), (73), (76), and (81) into Eq. (51) gives

$$\begin{aligned} G = \frac{V}{2} \Bigg[& \left(\sigma_0 - (1 - v_1)T\Delta_1 \right)^2 \frac{d}{dA}\left(\frac{1}{E_{xx}^*} \right) + 2(1 - v_1)T\Delta_2 \left(\sigma_0 - (1 - v_1)T\Delta_1 \right) \frac{d}{dA}\left(\frac{\nu_{xy}^*}{E_{xx}^*} \right) \\ & + (1 - v_1)^2 T^2 \Delta_2^2 \frac{d}{dA}\left(\frac{1}{E_{yy}^*} \right) \Bigg] \end{aligned} \tag{82}$$

The energy release rate depends only on E_{xx}^*, E_{yy}^*, and ν_{xy}^*, all of which can be evaluated by an isothermal stress analysis. Equation (82) is exact for any damage mode in a microcracked laminate. If the damage is mode I propagation of a new microcrack between the existing microcracks, we can use Eq. (31) to derive an exact relation between laminate effective properties:

$$\frac{\dfrac{d}{dA}\left(\dfrac{\nu_{xy}^*}{E_{xx}^*}\right)}{\dfrac{d}{dA}\left(\dfrac{1}{E_{xx}^*}\right)} = \frac{\dfrac{d}{dA}\left(\dfrac{1}{E_{yy}^*}\right)}{\dfrac{d}{dA}\left(\dfrac{\nu_{xy}^*}{E_{xx}^*}\right)} \tag{83}$$

Interestingly, an analysis of fracture in the presence of residual thermal stresses has lead to a relation between effective mechanical properties. Using Eq. (83) the mode I microcracking energy release is

$$G_I = \frac{V}{2}\frac{\left[\left(\sigma_0 - (1-v_1)T\Delta_1\right)\dfrac{d}{dA}\left(\dfrac{1}{E_{xx}^*}\right) + (1-v_1)T\Delta_2\dfrac{d}{dA}\left(\dfrac{\nu_{xy}^*}{E_{xx}^*}\right)\right]^2}{\dfrac{d}{dA}\left(\dfrac{1}{E_{xx}^*}\right)} \tag{84}$$

The mode I energy release rate depends only on E_{xx}^* and ν_{xy}^*. McCartney (1994) derived results that appear equivalent to Eqs. (82) and (83). McCartney's results, however, were a specific result of an approximate generalized, plane-strain, thermal elasticity analysis of a microcracked laminate. The results here were derived by a very different path and are exact provided exact results for E_{xx}^*, ν_{xy}^*, and E_{yy}^* are known.

The previous results were for uniaxial loading. It is trivial to extend the analysis to generalized in-plane loading of the microcracked laminate. The energy release rate becomes

$$G = \frac{V}{2}\boldsymbol{\sigma}^{eff} \cdot \frac{d}{dA}\begin{pmatrix} \dfrac{1}{E_{xx}^*} & -\dfrac{\nu_{xy}^*}{E_{xx}^*} & 0 \\ -\dfrac{\nu_{xy}^*}{E_{xx}^*} & \dfrac{1}{E_{yy}^*} & 0 \\ 0 & 0 & \dfrac{1}{G_{xy}^*} \end{pmatrix}\boldsymbol{\sigma}^{eff} \qquad \boxed{\begin{array}{c}\text{Note: } V/2 \text{ is a}\\ \text{correction } vs.\\ \text{published paper}\end{array}} \tag{85}$$

where G_{xy}^* is the effective in-plane shear modulus and $\boldsymbol{\sigma}^{eff}$ is the effective in-plane applied stressed defined by

$$\boldsymbol{\sigma}^{eff} = \left(\sigma_{x0} - (1-v_1)T\Delta_1, \sigma_{y0} - (1-v_1)T\Delta_2, \tau_{xy0}\right) \tag{86}$$

Here σ_{x0}, σ_{y0}, and τ_{xy0} are the applied in-plane stresses.

Discussion

In many examples considered, the energy release rate for crack growth, G, was reduced to an expression that depends only on the effective compliance ($\boldsymbol{S}^*$) or stiffness ($\boldsymbol{C}^*$) tensors and the phase average mechanical stresses. Because these terms can be evaluated by isothermal elasticity methods (Hashin 1963; 1969), it is often possible to include the effect of residual thermal stresses on the energy release rate without ever evaluating the thermal stresses. In other words, no thermal elasticity analysis of the composite is required. This result, however, does not imply that the effect of residual thermal stresses is a trivial extension of isothermal fracture mechanics analysis. For example, the isothermal energy release rate for microcracking of a laminate depends only on the E_{xx}^*. In contrast, the energy release rate in the presence of residual stresses depends on E_{xx}^*, ν_{xy}^*, and E_{yy}^*. Thus, in general, the fracture mechanics analysis of a composite with residual thermal stresses will require more isothermal elasticity analysis of composite properties than the corresponding isothermal fracture mechanics analysis.

The discussion here was confined to residual thermal stresses, but identical results could be derived for other types of residual stresses. For example, fiber-reinforced, polymer-matrix composites can develop residual stresses due to differential moisture absorption between the fibers and the matrix (Tsai, 1987). The results for moisture-induced residual stresses can be generated from the results given here by replacing $\boldsymbol{\alpha}$ by the moisture expansion tensor and T by the moisture concentration. Residual stresses induced by nonuniform

temperature (T a function of position) are more complicated. The general theory derived here up to Eq. (23) still applies when there is nonuniform T. Many subsequent simplifications, however, only hold for uniform T.

The energy release rate in the presence of residual stresses depends on derivatives of mechanical terms like

$$\frac{d\boldsymbol{S}^*}{dA}, \quad \frac{d\boldsymbol{C}^*}{dA}, \quad \text{and} \quad \frac{d\overline{\boldsymbol{\sigma}^{(im)}}}{dA} \tag{87}$$

Many damage mechanics models have been developed that give effective properties as a function damage. The results in this paper could easily be incorporated into such models to simultaneously give the energy release rate for damage growth. An important observation, however, is that the ability of a damage analysis to predict effective properties is *necessary* but not *sufficient* justification for use of that analysis to calculate energy release rate. The problem is that calculating derivatives of effective properties is a harder problem than calculating the average property itself. Analyses that are accurate to within a few percent for effective properties may be grossly in error when calculating the derivative. Thus any damage mechanics analysis that seeks to calculate energy release rate must be verified to have accurate *derivatives* of the required terms. Experimentally this verification can be done by comparing fracture experiments to fracture mechanics predictions. In contrast, experimental verification based on effective property predictions, such as modulus reductions, are inadequate tests of energy release rate models.

Acknowledgments

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