A BESSEL FUNCTION ANALYSIS OF STRESS TRANSFER FROM THE MATRIX TO THE FIBER THROUGH AN IMPERFECT INTERFACE: APPLICATION TO THE FRAGMENTATION TEST

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ABSTRACT

The problem of stress transfer from the matrix to the fiber is central to understanding the performance of composite materials. There have been numerous approaches to analyzing stress transfer (*e.g.*, shear-lag or finite element analysis); this paper will discuss a new approach using Bessel functions. In brief, the new analysis is exact (except near the fiber ends), accounts for thermal stresses, can handle anisotropic fibers, and, most importantly, can describe stress transfer through an imperfect interface. Some analysis results will be discussed. Those results will then be used to interpret results from the single-fiber fragmentation test.

INTRODUCTION

In the single-fiber fragmentation test, a single fiber is embedded in a matrix and pulled in tension until the fiber fractures into many fragments [1-2]. The average value of the distribution of fragment sizes existing after the fragmentation processes ceases is used to measure interfacial toughness. The fragmentation test can be interpreted as a direct observation of stress transfer from the matrix through the interface or interphase and into the fiber. If the interface is good, stress transfer will be rapid and the final fragment lengths will be short. Conversely, if the interface is poor, the final fiber fragments will be long. The stress transfer process is influenced by the mechanical properties of the matrix, the fiber, and the interphase, by whether the interface is intact or debonded, and by whether the interphase region is elastic, plastic, or viscoelastic.

The fragmentation test is normally analyzed by a simple elastic-plastic analysis [3]. The interfacial shear stress is assumed to be elastic until failure at which point it becomes a constant interfacial yield stress called the interfacial yield strength, τ_c. The critical fiber fragment length, l_c, is defined as the smallest length of fiber for which the post-yield interfacial yield stress can transfer a tensile stress back into the fiber that is equal to the strength of the fiber. A simple equilibrium analysis gives

$$\tau_{ic} = \frac{\sigma_f(l_c) r_f}{l_c} \qquad (1)$$

where $\sigma_f(l_c)$ is the tensile strength of the fiber of radius r_f whose length is equal to l_c. The inadequacy of Eq. (1) can be demonstrated in many ways. The most direct is to check the assumption of constant interfacial shear stress along the fiber fragment. Experimental results using Raman spectroscopy show that the interfacial shear stress is not constant [4].

To get a better understanding of the fragmentation test, we need a better analysis of stress transfer through the interface and into the fiber. To be able to explain the effect of the interface, that analysis must be able to describe stress transfer across an imperfect interface. The relevant stress analysis problem is illustrated in Fig. 1. Figure 1B shows a single fiber fragment in an infinite matrix. Figures 1C and 1D split the problem into two parts. Figure 1C is the far field stresses. These can easily be solved by noting zero

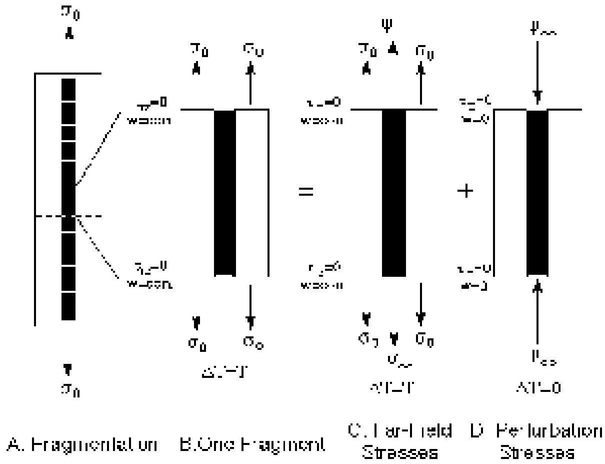

Fig. 1: Fragmentation test stress analysis boundary conditions. σ_0 is the applied stress, ΔT is the thermal load, and σ_∞ is the far-field stress in the fiber.

shear stress and constant axial strain in both components [5]. The difficult problem is in Fig. 1D. We thus consider the stresses in the fiber and the matrix for a compressive stress on the fiber ends and zero displacement and shear stress on the matrix ends.

Despite the importance of stress transfer, there are surprisingly few analytical methods that can be used to study the role of the interface in the stress transfer process. Many investigators use shear-lag methods [6]. While shear-lag methods give a useful qualitative description of the stress transfer process, they are of little help in developing quantitative models. They include an unknown parameter (the shear-lag parameter) and because they are one-dimensional, they ignore the potentially important effects of radial stresses. Various more sophisticated analyses have also been proposed [7,8], but they always deal with isotropic fibers and typically deal with different boundary conditions. The more common problem solved is with zero traction on the matrix ends rather than zero displacement. The zero displacement problem, however, is the relevant one for the fragmentation test.

In this paper, I will describe a new analysis specific for the fragmentation test using Bessel functions. The analysis is exact, except near the fiber ends, and can describe stress transfer through an imperfect interface. The analysis will be used to give some predictions of stress transfer. Finally, the analysis will be used to model fragmentation test results.

BESSEL FUNCTION ANALYSIS

Stress-function analyses using Bessel-Fourier series have historically be used to find axisymmetric stress states in cylinders (*i.e.*, fibers) with arbitrary transverse loading (*i.e.*, radial and shear stresses at the interface). The approach works for both isotropic fibers [9] and transversely isotropic fibers [10]. Space limitation precludes developing the stress function in this paper. We merely quote a fiber and matrix stress function that satisfies the boundary conditions in Fig. 1D and therefore can be used, with the help of formulas in Ref. [10], to calculate all stresses, strains, and displacements. In the fiber, the stress function is

$$\phi(r,z) = B_1 z^3 + B_2 r^2 z + B_3 \left(\frac{15}{2} a d r^3 z - 10 d r z^3\right) + \sum_{i=1} \sin k_i z \left(b_{1i} I_0(\lambda_{1i} r) + b_{2i} I_0(\lambda_{2i} r)\right) \quad (2)$$

In the matrix, the stress function is

$$\phi(r,z) = A_1 z \ln r + \sum_{i=1} \sin k_i z \left(a_{0i} K_0(k_i r) + a_{1i} k_i r K_1(k_i r)\right) \quad (3)$$

where $I_0(r)$, $K_0(r)$, and $K_1(r)$ are modified Bessel functions of the first and second kind, a and d depend only on the mechanical properties of the fiber [10], and k_i, λ_{1i}, and λ_{2i}, depend only on the fragment length and the mechanical properties of the fiber. The remaining terms (A_1, B_i, a_{ji}, and b_{ji}) are unknown and need to be determined. The only approximation in this stress function is that the stress on the fiber end is not uniformly equal to $-\sigma$. Instead, the average tensile stress on the fiber end is equal to $-\sigma$.

The unknown constants can be determined by using constraints imposed by the fiber/matrix interface. Regardless of the state of the interface (perfect or imperfect, bonded or debonded), equilibrium requires the interfacial shear stresses and radial stresses to be continuous at the interface. For more constraints, we look to the interfacial displacements. For an imperfect interface, we follow Hashin and allow a discontinuity in the axial and radial displacements at the interface (w(r) and u(r) at $r=r_f$) [11]. Hashin further suggested that the axial displacement discontinuity is proportional to the interfacial shear stress and the radial displacement discontinuity is proportional to the interfacial radial stress [11]. Thus

$$(w_f(r_r) - w_m(r_f))/r_f = \tau_{rz}(r_f)/D_s \quad (4)$$

$$(u_f(r_f) - u_m(r_f))/r_f = \tau_{rr}(r_f)/D_n \quad (5)$$

where D_s and D_n are interface parameters describing the quality of the interface (note that the displacement discontinuities have been normalized to r_f, making the units of D_s and D_n units of a modulus). As the interface parameters approach , the interface becomes perfect (*i.e.*, no displacement discontinuities). As the interface parameters approach zero, the interface becomes debonded (*i.e.*, zero interfacial stress). An important simplification for the fragmentation test is that the interfacial radial stresses are compressive. Because negative displacement discontinuities in the radial direction are highly unlikely, D_n must be set to . With this fortunate simplification, the imperfect interface is described by a single parameter, D_s. Finally, the two stress and two displacement conditions at the interface provide exactly the correct number of equations for determining all the constants in the stress functions. The numerical overhead is minimal, requiring solution of a 4X4 linear system for each Fourier term included in the stress function. The entire analysis is easily accomplished on a PC.

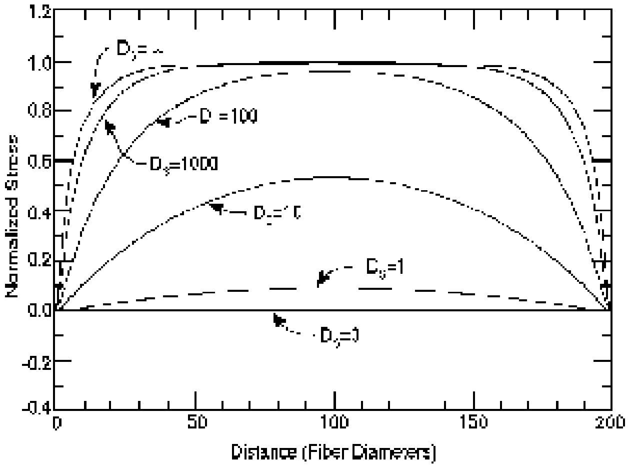

Fig. 2: Effect of D_s on stress transfer from the matrix to the fiber for a carbon fiber fragment of axial ratio 200 embedded in an epoxy matrix. The axial fiber stress is normalized to the far-field fiber stress.

RESULTS

Figure 2 plots the axial stress in a typical carbon fiber embedded in an epoxy matrix as a function of distance from the fiber end. The various curves show the effect of the interface parameter D_s. An infinite D_s (perfect interface) shows the stress transferring back into the fiber in about 30 to 50 fibers diameters. As shown below, this stress transfer rate agrees with Raman spectroscopy results. As D_s decreases, the stress transfer distance increases. Eventually when D_s is zero, the interface is debonded and no stress is transferred to the fiber.

Figure 3 compares the theoretical predictions to Raman spectroscopy results (from Ref. [4]) for an isolated fiber break at low strain. There is significant scatter in the experimental results, but the perfect interface model does a good job of predicting the rate of stress transfer. A slightly lower value for D_s (500 MPa) does a quantitative job of predicting stress transfer.

At higher strain during a fragmentation test, the fiber fragments become shorter and there is probably interfacial damage near the fiber breaks. A comparison of predicted and measured stress transfer in a short fragment at high strain (from Ref. [4]) is given in Fig. 4. No single value of D_s can predict all the experimental data. Instead, we suggest

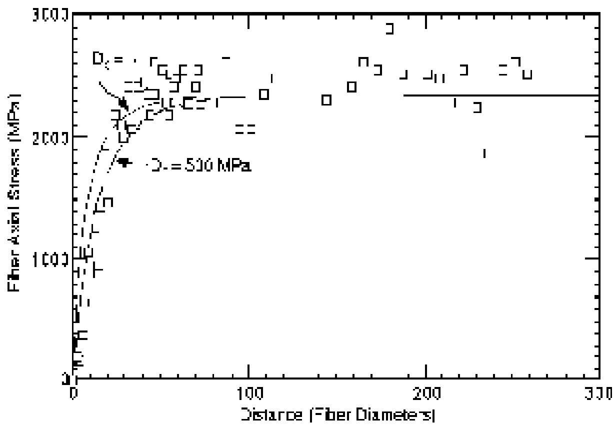

Fig. 3: Raman spectroscopy measurement of the axial stress in the fiber for an isolated fiber break at low strain. Solid lines are theoretical predictions.

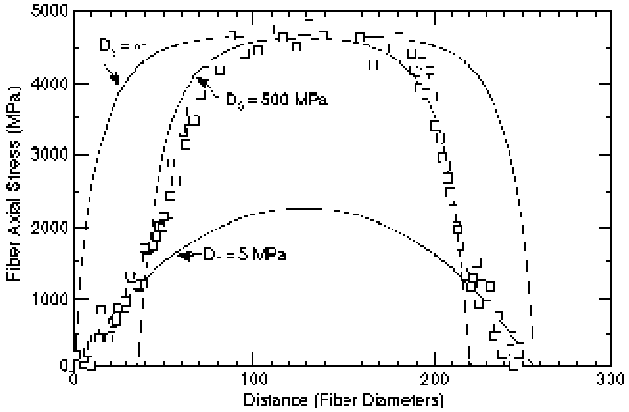

Fig. 4: Raman spectroscopy measurement of the axial stress in the fiber for an isolated fiber fragment at high strain. Solid lines are theoretical predictions.

that the damage near the fiber ends causes D_s to be low. In the middle of the fiber there is no interfacial damage and D_s remains high. A D_s value of 5 MPa near the fiber ends and 500 MPa in the middle (as used in Fig. 3) can quantitatively fit all the stress transfer data.

For a given value of D_s we can calculate the axial stress in the fiber for a fragment of any length. To model a fragmentation test, we assume that the fiber length-strength relation follows a Weibull distribution. Because the highest stress is always in the middle of a fiber fragment, we assume the next fiber break will occur when the stress in the middle of the fiber fragment reaches the strength of that fragment. The simple expression is

$$<\sigma_{zz,f}(z=0)> = \sigma_f l^{-1/m}\Gamma(1 + 1/m) \qquad (6)$$

where σ_f and m are the Weibull scale and shape parameters for the fiber, $\Gamma(x)$ is the Gamma function, and $<\sigma_{zz,f}(z=0)>$ is the average axial fiber tensile stress stress in the middle of the fiber fragment calculated using the Bessel function analysis. Numerically solving Eq. (6) for fragment length l as a function of applied stress gives a prediction of a fragmentation test.

Figure 5 plots typical fragmentation data for an AS4 carbon fiber in an epoxy matrix. The results are plotted as fiber break density (1/fragment length) as a function of applied strain. The prediction using D_s = 1000 is consistent with the data. At high strain, there is an indication of the experimental data reaching a plateau while the theoretical prediction continues to rise. We suggest that a single value of D_s is an oversimplification. As suggested by the results in Fig. 4, the high strain data will probably require a two-zone model with the zone near the fiber end having a lower D_s. This extra work required to fit experimental results has physical justification. The higher strains damage the interface thus making it more imperfect and the local D_s lower.

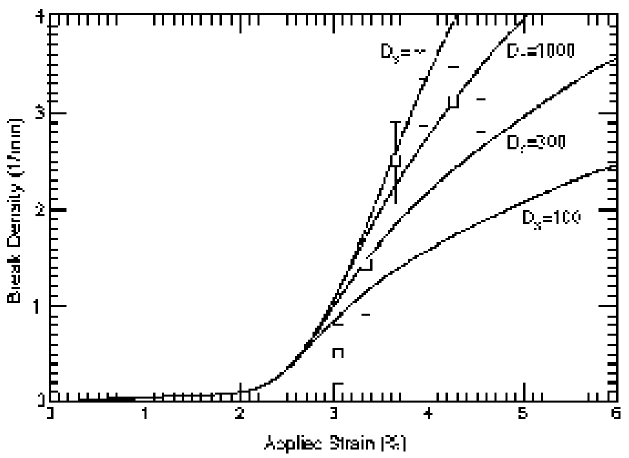

Fig. 5: Break density as a function of applied strain for an AS4 carbon fiber embedded in an epoxy matrix.

CONCLUSIONS

The Bessel function analysis provides a new tool for predicting stress transfer from the matrix to the fiber. Although a single value of D_s can not predict all experimental results, a simple generalization of Eq. (4) to

$$w_f(r_f) - w_m(r_f) = \frac{\sigma_{rz}(r_f)}{D_s(\sigma_{rz}(r_f),\sigma_{rr}(r_f),\sigma_{\theta\theta}(r_f))} \qquad (7)$$

works well. In other words, the value of D_s becomes a function of the stress state at the interface. At low strains, the interface is nearly perfect and D_s is large. High strains near a fiber break, however, can cause the interface to become damaged and D_s to drop to a low value.

Figure 5 shows that the fragmentation test can be analyzed using the Bessel function analysis. The end result of a fragmentation test as a function of applied strain is a measurement of D_s. With more accurate data it should be possible to measure D_s before and after interfacial damage. The advantage of D_s over the now commonly measured interfacial shear strength is that D_s has the potential of being useful in making predictions about real laminates. Hashin has outlined a method for calculating laminate properties as a function of fiber and matrix properties and interface parameters such as D_s [11]. Thus for the first time, experimental results from single-fiber specimens could be useful for making quantitative predictions about real laminates.

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