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SOME NEW VARIATIONAL MECHANICS RESULTS ON COMPOSITE MICROCRACKING

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ABSTRACT

Using variational mechanics we present mathematical bounds on the effective modulus of a cross-ply laminate containing microcracks in the 90° plies. An average of the two bounds provides an effectively exact solution to the modulus reduction problem. The problem of predicting microcracking is reduced to finding the energy release rate for formation of a complete microcrack. From the bounds on effective modulus we derive both rigorous and practical bounds to the energy release rate for microcracking. Again, an average of the two bounds provides an effectively exact solution to the energy release rate problem.

INTRODUCTION

The first form of failure in laminates containing off-axis plies is generally microcracking of those plies. There is a long literature on predicting the formation of microcracks and on predicting the effect they have on the mechanical properties of a laminate. It is beyond the scope of this paper to review that literature. The reader is referred instead to a recent review article in Ref. [1]. The goal here is to calculate the effective modulus, E_A^*, and the energy release rate for formation of a new microcrack, G_m, both as a function of the microcrack density. The problem is analyzed using some new variational mechanics results. These results permit establishing mathematically rigorous bounds on both E_A^* and G_m. We find that an average of the rigorous bounds provides an effectively exact solution to both the modulus reduction and the energy release rate problems.

FRACTURE MECHANICS ANALYSIS

Imagine a microcrack initiating somewhere between the two existing microcracks (see Fig. 1) and propagating in the width direction (y direction). The energy release rate for propagation of a microcrack of current area A is

$$G_m(A) = \frac{\partial U_{ext}}{\partial A} - \frac{\partial U}{\partial A} \tag{1}$$

where U_{ext} is external work and U is total strain energy. Some recent three-dimensional finite element analyses have shown that $G_m(A)$ is virtually independent of the length of the propagating microcrack [2, 3]. By this fortunate simplification, we can calculate $G_m(A)$ by analyzing the energy released due to the formation of a *complete* microcrack, denoted as G_m:

$$\frac{1}{2t_1W}\int_0^A G_m(A)\,dA = G_m(A) = G_m = \frac{\Delta U_{ext} - \Delta U}{2t_1W} \tag{2}$$

where ΔU_{ext} and ΔU are the total external work and change in stain energy associated with formation of a *complete* microcrack, and $2t_1W$ is the fracture area of a single microcrack (see

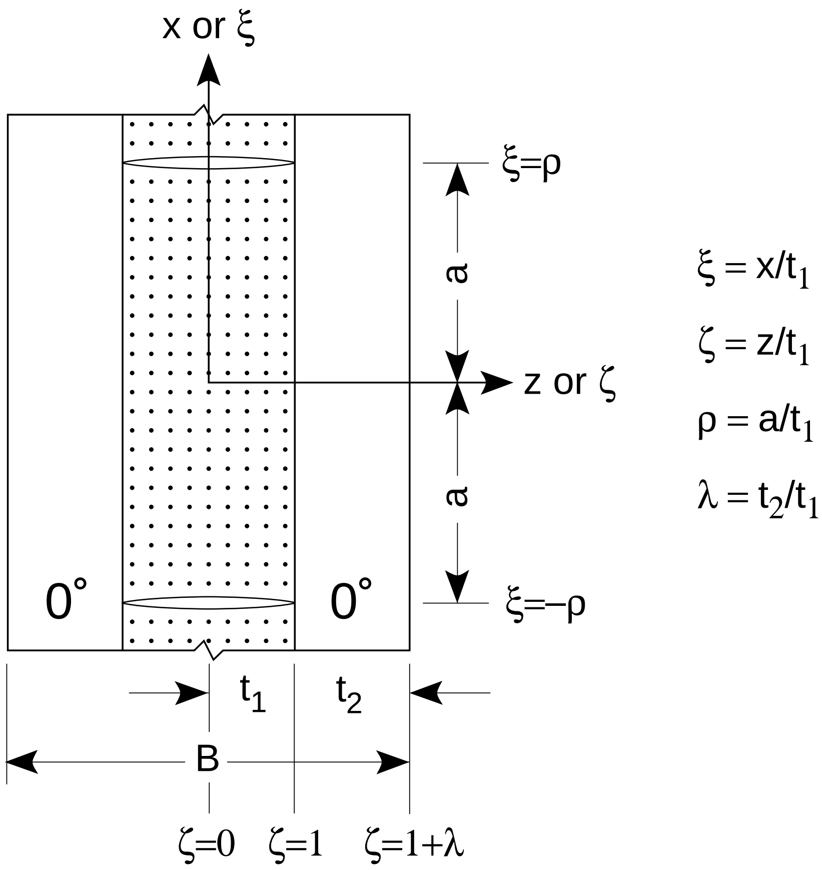

Fig. 1. A coordinate system on the edge (x-z plane) of a cross-ply laminate with two microcracks located at $x = \pm a$. The y direction is the width direction of the laminate; the laminate width is W. ξ and ζ are dimensionless coordinates in the x and y directions.

Fig. 1). Many literature results have shown that the formation of new microcracks can be predicted by assuming they form when G_m reaches G_{mc} or the critical microcracking toughness for the laminate [1, 4]. This "energy" analysis of microcracking has sometimes been criticized as not being rigorous fracture mechanics because it does not involve analyzing incremental growth of an existing crack. The equivalence of G_m and $G_m(A)$, however, argues that it is rigorous fracture mechanics. In fact, the connection with fracture mechanics is probably the reason the energy analysis works well [4]. We claim that the problem of predicting microcracking in $[0_n/90_m]_s$ laminates is equivalent to accurately finding G_m.

We consider formation of a complete microcrack due to an axial stress of σ_0 and thermal load T. The total axial displacement between the two existing microcracks can be written as

$$u = 2a\left(\frac{\sigma_0}{E_A^*} + \alpha_A^* T\right) \tag{3}$$

where E_A^* and α_A^* are the effective axial modulus and thermal expansion coefficient of the laminate. The external work associated with the formation of a new microcrack between the existing microcracks is $\Delta U_{ext} = \sigma_0 BW\Delta u$ where B is the laminate thickness. For strain energy, we imagine a sample under arbitrary traction loads but no thermal load ($T = 0$). The tractions will result in some stresses and strains in the body denoted $\vec{\sigma_m}$ and $\vec{\varepsilon_m} = S\vec{\sigma_m}$ where S is the compliance tensor. Now change the thermal load to T without changing the surface tractions. The stresses will change to $\vec{\sigma} = \vec{\sigma_m} + \vec{\sigma_r}$ where $\vec{\sigma_r}$ are the residual thermal stresses. The total strain energy is

$$U = \int_V \frac{1}{2}\vec{\sigma}\cdot S\vec{\sigma}dV = \int_V \frac{1}{2}\vec{\sigma_m}\cdot S\vec{\sigma_m}dV + \int_V \vec{\varepsilon_m}\cdot\vec{\sigma_r}dV + \int_V \frac{1}{2}\vec{\sigma_r}\cdot S\vec{\sigma_r}dV \tag{4}$$

Because the residual stresses, by themselves, obey equilibrium and induce no new surface tractions, the principles of virtual work imply that the cross term between mechanical strain and residual stresses is zero. Thus total total strain energy can be written as a sum of mechanical strain energy and thermal strain energy. In terms of effective modulus

$$U = 2aBW\left(\frac{1}{2}\frac{\sigma_0^2}{E_A^*} + U_{res}\right) \tag{5}$$

where U_{res} is the residual strain energy per unit volume. Substituting Eqs. (3) and (5) into Eq. (2) gives

$$G_m = \rho B\left[\frac{\sigma_0^2}{2}\Delta\left(\frac{1}{E_A^*}\right) + \sigma_0 T\Delta\alpha_A^* - \Delta U_{res}\right] \tag{6}$$

where ρ is a dimensionless crack spacing.

We can eliminate α_A^* by using a remarkable theorem by Levin [5], which when applied to uniaxial loading states that

$$\alpha_A^*\sigma_0 = \frac{1}{V}\int_V \vec{\alpha}\cdot\vec{\sigma}\,dV \tag{7}$$

where $\vec{\sigma}$ are the stresses that result from a purely mechanical loading (*i.e.* $T = 0$) of a stress σ_0 in the axial direction. Defining a phase average stress in phase i as

$$\overline{\sigma_{jk}^{(i)}} = \frac{1}{V_i}\int_{V_i}\sigma_{jk}^{(i)}dV \tag{8}$$

where V_i is the volume of phase i, Levin's theorem for axial loading of $[0_n/90_m]_s$ laminates reduces to

$$\sigma_0\alpha_A^* = \alpha_T\sigma_0 - \frac{\Delta\alpha\lambda\overline{\sigma_{xx}^{(2)}}}{1+\lambda} \tag{9}$$

where $\Delta\alpha = \alpha_T - \alpha_A$ is the difference between the axial thermal expansion coefficients of the 90° plies (α_T) and the 0° plies (α_A), and $\lambda = t_2/t_1$. Here and elsewhere, a subscript 1 or superscript (1) denotes a property of the 90° plies; likewise a 2 denotes a property of the 0° plies. Now, in cross-ply laminates, the effective modulus is equivalent to the net stress divided by the phase average strain in the uncracked 0° plies: $E_A^* = \sigma_0/\overline{\varepsilon_{xx}^{(2)}} = E_A\sigma_0/\overline{\sigma_{xx}^{(2)}}$ where E_A is the axial modulus of the 0° plies. We quickly derive

$$\alpha_A^* = \alpha_0 - \frac{\Delta\alpha}{C_{1L}E_T}\left(\frac{E_0}{E_A^*} - 1\right) \tag{10}$$

where

$$E_0 = \frac{E_T + \lambda E_A}{1+\lambda} \qquad \alpha_0 = \frac{\alpha_T E_T + \lambda\alpha_A E_A}{(1+\lambda)E_0} \qquad C_{1L} = \frac{1}{E_T} + \frac{1}{\lambda E_A} = \frac{(1+\lambda)E_0}{\lambda E_A E_T} \tag{11}$$

E_0 and α_0 are the rule-of-mixtures axial modulus and axial thermal expansion coefficient of the uncracked laminate and E_T is the axial modulus of the 90° plies. Substituting Eq. (10) into Eq. (6) gives

$$G_m = \rho B\left[\left(\frac{\sigma_0^2}{2} - \frac{\sigma_0\Delta\alpha T E_0}{C_{1L}E_T}\right)\Delta\left(\frac{1}{E_A^*}\right) - \Delta U_{res}\right] \tag{12}$$

The fracture mechanics problem of microcracking is thus reduced to finding accurate results, or perhaps bounds, for E_A^* and U_{res}.

VARIATIONAL MECHANICS

Hashin [6] used an assumed stress state for a cross-ply laminate and minimized complementary energy to get the lower bound modulus, E_A^L:

$$\frac{1}{E_A^*} \le \frac{1}{E_A^L} = \frac{1}{E_0} + \frac{E_T^2}{E_0^2}\frac{C_{3L}}{1+\lambda}\frac{\chi_L(\rho)}{\rho} \tag{13}$$

The constant C_{3L} and the function $\chi_L(\rho)$ depend only on the geometry of the layup and on mechanical properties of the plies. To save space, the reader is referred to Ref. [1] for definitions of all the constants and functions in Hashin's analysis. Nairn extended Hashin's analysis to included residual stresses [7]. The total strain energy can be written as

$$U = 2aBW\left[\frac{\sigma_0^2}{2E_A^L} + \frac{\Delta\alpha^2 T^2}{2C_{1L}(1+\lambda)}\left(1 - \frac{C_{3L}}{C_{1L}}\frac{\chi_L(\rho)}{\rho}\right)\right] \tag{14}$$

As expected, the total strain energy is a sum of mechanical strain energy and residual strain energy. Using E_A^L to eliminate $\chi_L(\rho)$, the residual strain energy density becomes

$$U_{res} = \frac{\Delta\alpha^2 T^2}{2C_{1L}}\left[\frac{1}{1+\lambda} - \frac{E_0^2}{C_{1L}E_T^2}\left(\frac{1}{E_A^L} - \frac{1}{E_0}\right)\right] \tag{15}$$

The remainder of this section will present a new variational mechanics result for an upper bound modulus. Unfortunately the analysis is too long to fit within the current page limits. We resort to an outline of the analysis and quote sufficient results for calculations. A more detailed analysis will be given in a future publication [8]. The upper bound modulus is found by using an assumed displacement field and minimizing the potential energy. We assumed the following x and z direction displacements:

$$\frac{u^{(1)}}{t_1} = \psi_1\phi + a_1\xi \qquad \frac{w^{(1)}}{t_1} = (\psi_2 + a_2)\zeta \qquad \frac{u^{(2)}}{t_1} = \psi_1 + a_1\xi \qquad \frac{w^{(2)}}{t_1} = \psi_2 + a_2 + a_3(\zeta - 1) \tag{16}$$

Here ψ_i are unknown functions of the dimensionless axial coordinate $\xi = x/t_1$, ϕ is a function of the dimensionless axial coordinate $\zeta = z/t_1$, and the a_i are constants that may depend on crack spacing. The necessary boundary conditions are $\psi_1(\rho) = \rho(\varepsilon_0 - a_1)$ (ε_0 is global strain), $\psi_2(\rho) = a_4$ (a new unknown constant), $\phi(1) = 1$, and $\phi'(0) = 0$. $\phi(\zeta)$ describes the crack-opening displacement of the microcrack; explicitly, the semi-crack-opening displacement is $\delta(\zeta) = \rho(\varepsilon_0 - a_1)(1 - \phi(\zeta))$ with boundary conditions $\delta(1) = \delta'(0) = 0$. The total potential energy, including terms for residual stresses, that must be minimized is

$$\Pi = \int_V \left(\frac{1}{2}\vec{\varepsilon}\cdot C\vec{\varepsilon} - \vec{\alpha}T\cdot C\vec{\varepsilon}\right)dV \tag{17}$$

where C is the stiffness tensor. Inserting the assumed displacement field gives, after much algebra:

$$\begin{aligned}\Pi \;=\; t_1^2 W\int_{-\rho}^{\rho}\Big[& C_1\psi_1^2 + C_2\psi_2^2 + C_3\psi_1'^2 + C_4\psi_2'^2 + C_5\psi_1'\psi_2 + C_6\psi_1\psi_2' \\ &+ \left(2C_2\langle\phi\rangle(a_1 + a_2\nu_T - (1+\nu_T)\alpha_T T) + 2C_7(a_1 + a_3\nu_A' - (\alpha_A + \nu_A'\alpha_T)T)\right)\psi_1' \\ &+ 2C_2(a_1\nu_T + a_2 - (1+\nu_T)\alpha_T T)\psi_2 + C_2(a_1^2 + 2a_1a_2\nu_T + a_2^2 - 2(a_1+a_2)(1+\nu_T)\alpha_T T) \\ &+ C_7\left(a_1^2 + 2a_1a_3\nu_A' + a_3^2\frac{\nu_A'}{\nu_A} - 2a_1(\alpha_A + \nu_A'\alpha_T)T - 2a_3\left(\nu_A'\alpha_A + \frac{\nu_A'}{\nu_A}\alpha_T\right)T\right)\Big]d\xi\end{aligned} \tag{18}$$

where

$$\begin{aligned} C_1 &= G_T\langle\phi'^2\rangle & C_2 &= \tfrac{E_T}{1-\nu_T^2} & C_3 &= \tfrac{E_T}{1-\nu_T^2}\langle\phi^2\rangle + \tfrac{\lambda E_A}{1-\nu_A\nu_A'} \\ C_4 &= \tfrac{G_T}{3} + \lambda G_A & C_5 &= \tfrac{2\nu_T E_T}{1-\nu_T^2}\langle\phi\rangle & C_6 &= 2G_T(1-\langle\phi\rangle) \\ & & C_7 &= \tfrac{\lambda E_A}{1-\nu_A\nu_A'} & & \end{aligned} \tag{19}$$

where G_T and G_A are the shear modulus of the 90° and 0° plies, ν_T and ν_A are the Poisson's ratios of the 90° and 0° plies, $\nu_A' = \frac{\nu_A E_{zz}^{(2)}}{E_A}$, and $E_{zz}^{(2)}$ is the transverse modulus of the 0° plies. The angle brackets denote integration over ζ from 0 to 1. We assume at first that the function ϕ and the constants a_i are given. The potential energy can then be minimized to find ψ_i using the calculus of variations. The coupled Euler equations are

$$\psi_1'' - q_1\psi_1 - p_1\psi_2' = 0 \qquad \psi_2'' - q_2\psi_2 - p_2\psi_1' = \frac{C_2(\nu_T a_1 + a_2 - (1+\nu_T)\alpha_T T)}{C_4} \tag{20}$$

where

$$q_1 = \frac{C_1}{C_3} \qquad p_1 = -\frac{C_5 - C_6}{2C_3} \qquad q_2 = \frac{C_2}{C_4} \qquad p_2 = \frac{C_5 - C_6}{2C_4} \tag{21}$$

Assuming complex characteristic roots (which holds for the calculations in this paper), the solutions for ψ_i are:

$$\begin{aligned} \psi_1 &= p_1(\alpha A_1 - \beta A_2)\sinh\alpha\xi\cos\beta\xi - p_1(\beta A_1 + \alpha A_2)\cosh\alpha\xi\sin\beta\xi & (22)\\ \psi_2 &= [A_1(\alpha^2 - \beta^2 - q_1) - 2\alpha\beta A_2]\cosh\alpha\xi\cos\beta\xi \\ &\quad - [2\alpha\beta A_1 + A_2(\alpha^2 - \beta^2 - q_1)]\sinh\alpha\xi\sin\beta\xi - \nu_T a_1 - a_2 + (1+\nu_T)\alpha_T T & (23)\end{aligned}$$

where

$$\alpha = \frac{1}{2}\sqrt{2\sqrt{q_1 q_2} + q_1 + q_2 + p_1 p_2} \qquad \text{and} \qquad \beta = \frac{1}{2}\sqrt{2\sqrt{q_1 q_2} - q_1 - q_2 - p_1 p_2} \tag{24}$$

The constants A_1 and A_2 can easily be found from the boundary conditions on ψ_1 and ψ_2 at ρ.

The next step is to substitute these functions back into the potential energy and thereby find the total strain energy:

$$U = \int_V \frac{1}{2}(\vec{\varepsilon} - \vec{\alpha}T)\cdot C(\vec{\varepsilon} - \vec{\alpha}T)dV = \Pi + \int_V \frac{1}{2}\vec{\alpha}T \cdot C\vec{\alpha}T\,dV \tag{25}$$

This result can than be minimized with respect to a_i to find those constants. The result is

$$\begin{aligned} a_1 &= \frac{\varepsilon_0\chi_0(\rho) - E_T\varepsilon_0\langle\phi\rangle + E_T\alpha_T T(1 - \langle\phi\rangle)}{\chi_0(\rho) + E_T(1 - 2\langle\phi\rangle)} & (26)\\ a_2 &= -\nu_T a_1 + (1+\nu_T)\alpha_T T & (27)\\ a_3 &= -\nu_A\varepsilon_0 + (\nu_A\alpha_A + \alpha_T)T & (28)\\ a_4 &= -\frac{\rho(\varepsilon_0 - a_1)}{2C_4\chi_4(\rho)}\left(C_3\chi_2(\rho) + C_4\chi_3(\rho) + \frac{C_5 + C_6}{2}\right) & (29)\end{aligned}$$

where

$$\begin{aligned} \chi_0(\rho) &= \rho\left[C_3\chi_1(\rho) - \frac{\left(C_3\chi_2(\rho) + C_4\chi_3(\rho) + \frac{C_5+C_6}{2}\right)^2}{4C_4\chi_4(\rho)}\right] - C_7 & (30)\\ \chi_1(\rho) &= \frac{-\alpha\beta q_1(\coth\alpha\rho\cot\beta\rho + \tanh\alpha\rho\tan\beta\rho)}{\alpha(\alpha^2+\beta^2-q_1)\operatorname{csch}2\alpha\rho - \beta(\alpha^2+\beta^2+q_1)\csc 2\beta\rho} & (31)\\ \chi_2(\rho) &= \frac{p_1(\alpha^2+\beta^2)(\alpha\operatorname{csch}2\alpha\rho - \beta\csc 2\beta\rho)}{\alpha(\alpha^2+\beta^2-q_1)\operatorname{csch}2\alpha\rho - \beta(\alpha^2+\beta^2+q_1)\csc 2\beta\rho} & (32)\\ \chi_3(\rho) &= \frac{\left[(\alpha^2-\beta^2-q_1)^2 + 4\alpha^2\beta^2\right](\alpha\operatorname{csch}2\alpha\rho + \beta\csc 2\beta\rho)}{p_1\left[\alpha(\alpha^2+\beta^2-q_1)\operatorname{csch}2\alpha\rho - \beta(\alpha^2+\beta^2+q_1)\csc 2\beta\rho\right]} & (33)\\ \chi_4(\rho) &= \frac{-\alpha\beta(\alpha^2+\beta^2)(\coth\alpha\rho\tan\beta\rho + \tanh\alpha\rho\cot\beta\rho)}{\alpha(\alpha^2+\beta^2-q_1)\operatorname{csch}2\alpha\rho - \beta(\alpha^2+\beta^2+q_1)\csc 2\beta\rho} & (34)\\ & & (35)\end{aligned}$$

Substituting these constants back into the strain energy gives

$$U = 2aBW\left[\frac{1}{2}E_0(\varepsilon_0 - \alpha_0 T)^2 + \frac{\Delta\alpha^2 T^2}{2(1+\lambda)C_{1L}} - \frac{E_T^2}{2(1+\lambda)E_0}(\varepsilon_0 - \alpha_T T)^2\frac{\chi_U(\rho)}{\rho}\right] \tag{36}$$

where

$$\chi_U(\rho) = \frac{\rho E_0(1 - \langle\phi\rangle)^2}{\chi_0(\rho) + E_T(1 - 2\langle\phi\rangle)} \tag{37}$$

Unfortunately, the total strain energy cannot be analytically minimized with respect to the crack-opening displacement function $\phi(\zeta)$. By examination of finite element results, we made the reasonable assumption that $\phi(\zeta)$ is parabolic. Thus $\phi(\zeta) = A(\zeta^2 - 1) + 1$ or $\delta(\zeta) = A(1 - \zeta^2)$ where A is an unknown constant.

We can find the upper bound modulus from the total strain energy by setting $T = 0$. The result is

$$E_A^* \leq E_A^U = E_0 - \frac{1}{1+\lambda}\frac{E_T^2}{E_0}\frac{\chi_U(\rho)}{\rho} \tag{38}$$

To partition the total strain energy into mechanical strain energy and residual strain energy, we partition the global strain into mechanical strain and thermal strain: $\varepsilon_0 = \varepsilon_m + \alpha_A^* T$. Substitution into the strain energy result and making use of Eq. (10) gives

$$U = 2aBW\left[\frac{1}{2}E_A^U\varepsilon_m^2 + \frac{T^2}{2}\left(\frac{\Delta\alpha^2}{(1+\lambda)C_{1L}} + E_0(\alpha_A^* - \alpha_0)^2 - \frac{(\alpha_A^* - \alpha_T)^2}{1+\lambda}\frac{E_T^2}{E_0}\frac{\chi_U(\rho)}{\rho}\right)\right] \tag{39}$$

The total strain energy partitions as expected. Using upper bound modulus to eliminate $\chi_U(\rho)$ and again using Eq. (10) gives

$$U_{res} = \frac{\Delta\alpha^2 T^2}{2C_{1L}}\left[\frac{1}{(1+\lambda)} - \frac{E_0^2}{C_{1L}E_T^2}\left(\frac{1}{E_A^U} - \frac{1}{E_0}\right)\right] \tag{40}$$

Notice that this result is the same as U_{res} in the lower bound analysis except that E_A^U is used instead of E_A^L.

RESULTS AND DISCUSSION

Mathematical bounds on the modulus of a cracked cross-ply laminate are

$$\frac{\rho E_0^2(1+\lambda)}{\rho E_0(1+\lambda) + E_T^2 C_{3L}\chi_L(\rho)} = E_A^L \leq E_A^* \leq E_A^U = E_0 - \frac{1}{1+\lambda}\frac{E_T^2}{E_0}\frac{\chi_U(\rho)}{\rho} \tag{41}$$

We further define an average of the bounds as

$$E_A^{avg} = \frac{1}{2}\left(E_A^L + E_A^U\right) \tag{42}$$

E_A^L, E_A^U, and E_A^{avg} are compared to some finite element calculations on a $[0/90_2]_s$ E-glass/epoxy laminate in Fig. 2. The E_A^U calculations used $A = 3.2$ because that gave the minimum modulus for a crack density of 0.5 mm^{-1}. We note, however, that large changes in A caused only negligible changes in E_A^U. The bounds on the modulus are relatively tight. E_A^{avg} provides an effectively exact solution to the modulus reduction problem.

Previous comparisons in the literature between E_A^L and experimental results [1, 6], suggest that E_A^L is closer to the correct modulus than is implied by Fig. 2. When the goal is to assess the accuracy of the variational mechanics analysis, the preferred comparison is between theory and finite element calculations. The observation that experimental results may differ from finite element calculations suggests that either the experiments were inaccurate or that the assumptions of the mechanics analysis lack realism. For example, the mechanics analysis assumes there is a perfect interface between the 0° and 90° plies. If this assumption is wrong, the mechanics analysis might overestimate the true modulus. Future work on modulus reduction in cross-ply laminates should address ply interface issues.

Substituting U_{res} from either the lower or upper bound analysis, the energy release rate becomes

$$G_m = \frac{\rho B}{2}\frac{E_0^2}{E_T^2}{\sigma_{x0}^{(1)}}^2\Delta\left(\frac{1}{E_A^*}\right) \qquad \text{where} \qquad \sigma_{x0}^{(1)} = \frac{E_T\sigma_0}{E_0} - \frac{\Delta\alpha T}{C_{1L}} \tag{43}$$

$\sigma_{x0}^{(1)}$ is the total axial stress in the 90° plies in the uncracked laminate [1]. We consider the formation of a new microcrack midway between two existing microcracks. A rigorous upper bound to G_m,

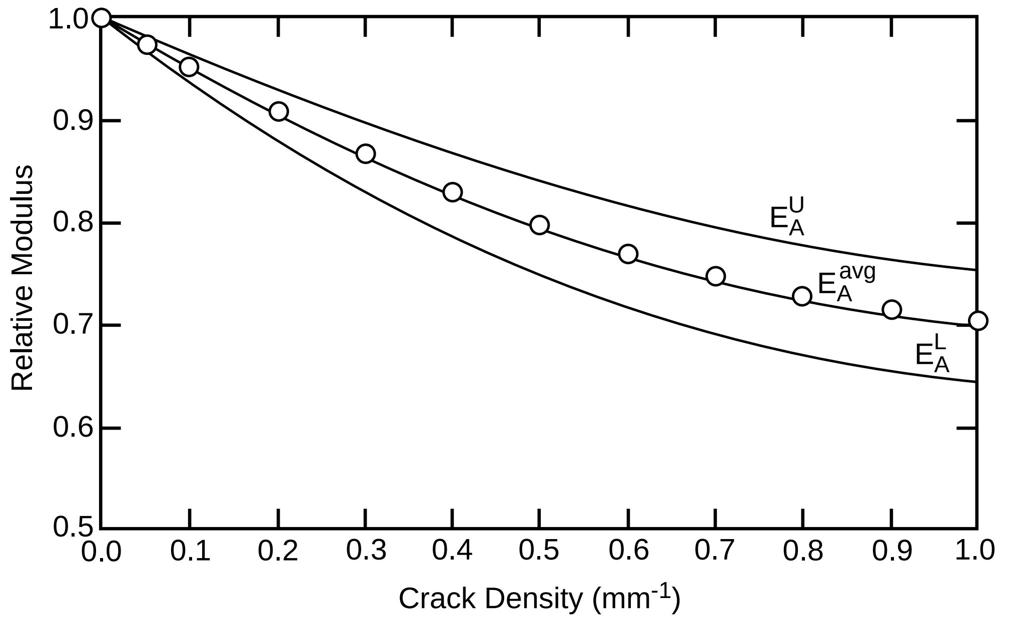

Fig. 2. Modulus reduction in a $[0/90_2]_s$ E-glass/epoxy laminate as a function of crack density. The calculation used $E_A = 41.7$ GPa, $E_T = 13$ GPa, $G_A = 3.4$ GPa, $G_T = 4.58$ GPa, $\nu_A = .30$, $\nu_T = .42$, $\alpha_A = 8.6 \times 10^{-6}$ $^\circ$C^{-1}, $\alpha_T = 22.1 \times 10^{-6}$ $^\circ$C^{-1}, $t_1 = 0.42$ mm, and $t_2 = 0.21$ mm.

denoted as G_m^U can be calculated by assuming the E_A^* is equal to E_A^L after the microcrack forms, but equal to E_A^U before the microcrack forms. Thus

$$G_m^U = \frac{\rho B}{2}\frac{E_0^2}{E_T^2}\sigma_{x0}^{(1)2}\left(\frac{1}{E_A^L(\rho/2)} - \frac{1}{E_A^U(\rho)}\right) \tag{44}$$

The converse assumption leads to a rigorous lower bound:

$$G_m^L = \frac{\rho B}{2}\frac{E_0^2}{E_T^2}\sigma_{x0}^{(1)2}\left(\frac{1}{E_A^U(\rho/2)} - \frac{1}{E_A^L(\rho)}\right) \tag{45}$$

The acknowledgment of the mathematical possibility of wild changes in E_A^* due to microcrack formation is probably overly pessimistic. Assume, for instance, that the correct modulus is at some specific position between E_A^L and E_A^U before microcrack formation. We claim that the correct modulus will likely remain close to the same relative position between E_A^L and E_A^U after the formation of a single new microcrack. If we accept this assumption, we can calculate new bounds on G_m which we refer to as "practical" bounds, as opposed to mathematically rigorous bounds. The practical bounds, denoted with superscript P, are given by the energy release rate calculated using only E_A^L or only E_A^U. Thus

$$G_m^{PU} = \frac{\rho B}{2}\frac{E_0^2}{E_T^2}\sigma_{x0}^{(1)2}\left(\frac{1}{E_A^L(\rho/2)} - \frac{1}{E_A^L(\rho)}\right) \qquad G_m^{PL} = \frac{\rho B}{2}\frac{E_0^2}{E_T^2}\sigma_{x0}^{(1)2}\left(\frac{1}{E_A^U(\rho/2)} - \frac{1}{E_A^U(\rho)}\right) \tag{46}$$

Finally, the success of E_A^{avg} suggests a similar calculation for energy release rate. Averaging either the rigorous bounds or the practical bounds gives

$$G_m^{avg} = \frac{\rho B}{2}\frac{E_0^2}{E_T^2}\sigma_{x0}^{(1)2}\left(\frac{1}{2E_A^L(\rho/2)} + \frac{1}{2E_A^U(\rho/2)} - \frac{1}{2E_A^L(\rho)} - \frac{1}{2E_A^U(\rho)}\right) \tag{47}$$

G_m^L, G_m^{PL}, G_m^{PU}, G_m^U, and G_m^{avg} are compared to some finite element calculations on a $[0/90_2]_s$ E-glass/epoxy laminate in Fig. 3. The rigorous lower bound is very low and of relatively little use. In fact it becomes negative which is physically impossible. The rigorous upper bound is about a factor of two larger than the finite element calculations. Although this bound is not that close to the true value, it might be useful in design. A design to avoid microcracking based on G_m^U would

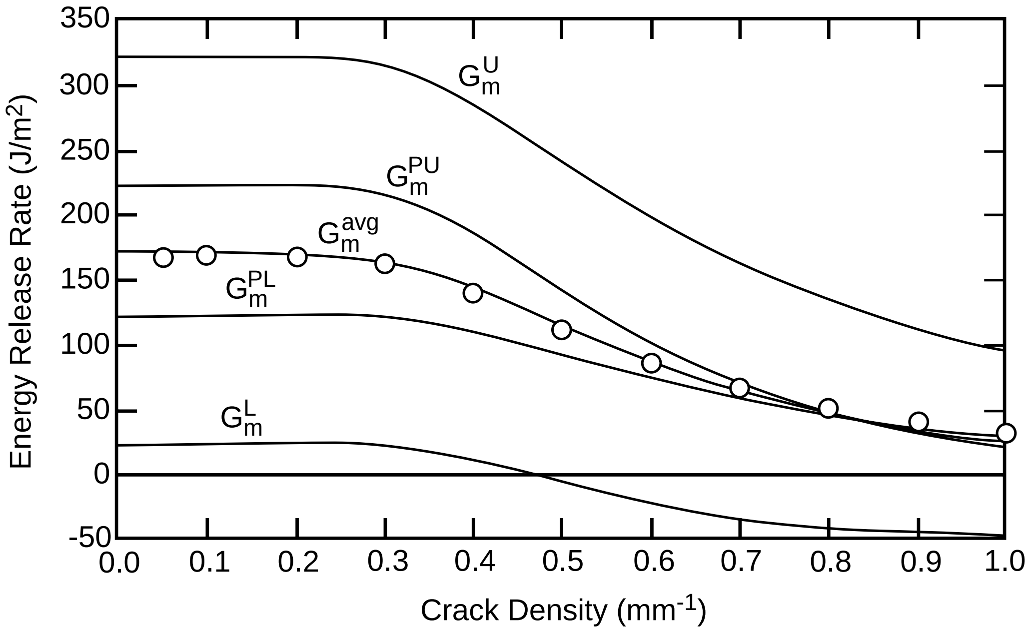

Fig. 3. Energy release rate for formation of a new microcrack in a $[0/90_2]_s$ E-glass/epoxy laminate as a function of crack density. The assumed laminate properties are given in the caption of Fig. 2. The calculations used $E_0\sigma_{x0}^{(1)}/E_T = 100\ MPa$.

have a built in safety factor. The practical bounds are much closer to the true energy release rate. The practical bounds are very tight at high crack density. This observation might explain why interpretation of experiments using G_m^{PU} tended to fit better at high crack densities than at low crack densities [1, 4]. Finally, G_m^{avg} provides an effectively exact solution to the energy release rate over the entire range of crack densities. We recommend using G_m^{avg} in future microcracking predictions in cross-ply laminates.

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